INTRODUCTION TO APPROXIMATION OF REAL CAUCHY PRINCIPAL VALUE (CPV) INTEGRALS

Singular integrals of the Cauchy principal type occur abundantly in solar physics, hydrodynamics and contour integration. Aitkenson (1978) defines singular integral in the following way which is quite general.

An integral (for simplicity one-dimensional integral) is said to be singular if the standard methods of numerical integration either do not apply or lead to slow convergence. This definition is from the point of view of Numerical Analysis.

However, from mathematical point of view, we say an integral to be singular if either of the following is true.

(i) The range of integration is infinite.

(ii) The integrand possesses infinite discontinuity at the end points of integration or at an intermediate point.

Both these types of singular integrals occur frequently in many branches of mathematical physics. It is noteworthy that the gamma function and beta function are singular integrals of the first and second types respectively. The following integral belongs to the category (ii).

$$F(g) = \int_a^b \frac{g(x)}{x-c}\, dx \tag{1}$$

where $a < c < b$ is of some special importance because of its application and usefulness in the theory of aerodynamics and scattering theory. The function g in equn. (1) is a well behaved function in $[a,b]$.

The integral $F(g)$ is said to be convergent if the following limit exist as ε and η which are small positive numbers tend to zero independently.

$$F(g) = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} \frac{g(x)}{x-c}\, dx + \lim_{\eta \to 0} \int_{c+\eta}^{b} \frac{g(x)}{x-c}\, dx \qquad (2)$$

If however, the above expression does not tend to a limit as ε and η tend to zero independently, it may still happen that

$$F(g) = \lim_{\varepsilon \to 0+} \left[\int_a^{c-\varepsilon} \frac{g(x)}{x-c}\, dx + \int_{c+\varepsilon}^{b} \frac{g(x)}{x-c}\, dx \right] \qquad (3)$$

may exist. When this is the case, we call this limit the Cauchy Principal Value (CPV) of the singular integral and denote it as

$$I(g,c) = P\int_a^b \frac{g(x)}{x-c}\, dx \quad or \quad I(g,c) = \int_a^b \frac{g(x)}{x-c}\, dx. \qquad (4)$$

In equn. (3) when the positive number ε tends to zero, if every contribution of the first integral is balanced by an equal and opposite contribution of the second integral inside the bracket, then the limit exists.

The exact evaluation of CPV integrals $I(g,c)$ presents immense difficulties as the situation involves limits. Hence, enough importance and emphasis has been laid on the numerical evaluation of CPV integrals. Some of the popular methods for numerical evaluation of $I(g,c)$ are due to Chawla and Jayarajan (1975), Hunter (1972), Iokamidis and Theocaris (1977), Elliott and Paget (1979), Monegato (1982), Paget and Elliott (1972), Price (1960), Theocaris and Kazantjakis (1981), Acharya and Das (1981) and so on.

The 9-point and 4-point rules due to Price (1960) and the 3-point Gauss rule due to Hunter (1972) are often used for their accuracy and simplicity for the numerical evaluation of the CPV integral

$$I(g,0) = P\int_{-1}^{1} \frac{g(x)}{x}\,dx \qquad (5)$$

It is pertinent to note that by a little modification of the integral $I(g,c)$, we can also use the above mentioned rules even if $c \neq 0$. This is done in the following way.

Let us consider the integral given by

$$I(g,c) = \int_{a}^{b} g(x)/(x-c)\;dx \qquad (6)$$

where $c \neq (a+b)/2$. Let $c \in (d,b)$ where $d=(a+b)/2$. Then, we can write

$$I(g,c) = \int_{a}^{\lambda} g(x)/(x-c)\,dx + \int_{\lambda}^{b} g(x)/(x-c)\,dx \qquad (7)$$

where $\lambda = 2c - b$. By this the point c is the central point of the line segment joining λ and b. It is noteworthy that the first integral in equn. (7) is a definite integral (since c is outside the range of integration) whereas the second integral is a CPV integral.

By suitable transformation, equn. (7) can be written in the following way.

$$I(g,c) = (2c-b-a)\int_{-1}^{1} \frac{g(((2c-b-a)t+2c+a-b)/2)}{(2c-b-a)t+a-b}\,dt + P\int_{-1}^{1}\frac{g(ht+c)}{t}\,dt \qquad (8)$$

where $h = b - c$ where of course $(a+b)/2 < c < b$.

Since the Price rules have 8th degree precision, we can apply Gauss-Legendre 4-point rule whose degree of precision is 9 for the approximation of the definite integral in equn. (8) and either of the Price rules for the approximation of the CPV integral in equn. (8), even though the point c is different from $(a+b)/2$. However, if we apply Hunter's 3-point rule for the approximation of the CPV integral in equn. (8), then the Gauss-Legendre 3-point rule can be applied for the numerical approximation of the definite integral in equn. (8).

CORRECTIVE FACTOR FOR THE RULES FOR REAL CPV INTEGRALS

It is known that presence of singularities near the path of integration affects the accuracy of the approximating rules. Lether (1977) has considered the following integral

$$I_1 = \int_{-1}^{1} \frac{e^x}{x^2 + 10^{-4}}\, dx \tag{9}$$

and the exact value of this integral correct to nine decimal places is equal to 313.172056239. The following table gives the value of the integral I_1 when evaluated by n-point Gauss rules.

Table 1

N	Approximate value
2	7.02
3	8891.32
4	13.24

The failure of the Gauss rules in case of the integral I_1 given by equn. (9) is solely due to the presence of nearby singularities *viz.* $z = \pm\, 10^{-2}\, i$ of the integrand in the integral. It is quite natural that presence of nearby singularities in case of the numerical evaluation of real CPV integrals also affects the accuracy of the approximating rules. One instance in this connection is the following CPV integral:

$$I_2 = P\int_{-1}^{1} \frac{e^x}{x(x^2+10^{-4})}\,dx \tag{10}$$

Acharya and Das (1981) have determined some corrective factors for the rules meant for the numerical computation of real CPV integral if nearby singularities of the integrand are present as is the case in example given in equn. (10).

Here the following real CPV integral is considered:

$$I(g,c) = P\int_{-1}^{1} \frac{g(x)}{x-c}\,dx \quad . \tag{11}$$

Let *b* be the only isolated singularity (pole) of order one of the complex function *g*(*z*), the analytic continuation of *g*(*x*), which is close to [-1,1]. Let further *p*(*z*) be the principal part of *f*(*z*) in the Laurent series expansion of the function *f*(*z*) in an annular region about the point $z = b$. The principal part of *g*(*x*) is given by

$$p(x) = \frac{h(b)}{x-b} \tag{11}$$

where $g(x)=h(x)(x\text{-}b)$ is obtained by factoring out $(x\text{-}b)^{-1}$ from *g*(*x*).

Then, φ(*x*) given by

$$\varphi(x) = g(x) - p(x) \tag{12}$$

is a regular function in the disk Ω centered about $x = 0$ of radius $r > 1$. Now equn. (12) implies that

$$I(g,c) = I(\varphi,c) + I(p,c). \tag{13}$$

If $R(\varphi,c)$ is a rule meant for the numerical approximation of the CPV integral $I(\varphi,c)$, then from the above equation, we have the following approximation

$$I(g,c) \approx R(\varphi,c) + I(p,c). \tag{14}$$

Using equn. (11) in $I(p,c)$, the following is obtained after simplification.

$$I(p,c) = \frac{h(b)}{c-b} \log\left\{\frac{(1-c)(b+1)}{(1+c)(b-1)}\right\} \tag{15}$$

consequently the equn. (14) boils down to

$$R(g,c) = \frac{h(b)}{c-b} \log\left\{\frac{(1-c)(b+1)}{(1+c)(b-1)}\right\} + R(\varphi,c) \tag{16}$$

If the m simple poles of $g(z)$ which are close to [-1,1] are b_1, b_2, ..., b_m then the approximation $R(g,c)$ to the CPV integral $I(g,c)$ is obtained by the generalization of equn. (16) in the following form.

$$R(g,c) = \sum_{j=1}^{m} \frac{h(b_j)}{c-b_j} \log\left\{\frac{(1-c)(b_j+1)}{(1+c)(b_j-1)}\right\} + R(\psi,c) \tag{17}$$

where the function $\psi(x)$ is obtained by subtracting out the principal parts corresponding to b_j , j=1(1)m from $g(x)$ *i.e.*

$$\psi(x) = g(x) - \sum_{j=1}^{m} \frac{h(b_j)}{c-b_j} \tag{18}$$

and the value $h(b_j)$ is obtained by evaluating $g(x)$ ignoring the factor x-b_j in the denominator of it.

In particular if

$$g(x) = \frac{h(x)}{x^2 + \varepsilon^2} \tag{19}$$

where ε is a small positive number, then the two nearby singularities of $g(z)$ are simple poles $b_1 = i\varepsilon$ and $b_2 = -i\varepsilon$. Then the desired approximation to the real CPV integral $I(g,c)$ is given by

$$R(g,c) = \frac{h(i\varepsilon)}{c - i\varepsilon}\log\left\{\frac{(1-c)(i\varepsilon+1)}{(1+c)(i\varepsilon-1)}\right\} + \frac{h(-i\varepsilon)}{c+i\varepsilon}\log\left\{\frac{(c-1)(1-i\varepsilon)}{(c+1)(1+i\varepsilon)}\right\} + R(\psi,c) \tag{20}$$

where

$$\psi(x) = \frac{1}{x^2+\varepsilon^2}\left\{g(x) - \frac{(x+i\varepsilon)g(i\varepsilon) - (x-i\varepsilon)g(-i\varepsilon)}{2i\varepsilon}\right\} \tag{21}$$

$$h(i\varepsilon) = \frac{g(i\varepsilon)}{2i\varepsilon}, \quad h(-i\varepsilon) = \frac{g(-i\varepsilon)}{2i\varepsilon}. \tag{22}$$

Most of the real CPV integrals with nearby singularities are of the form

$$I\left(\frac{g(x)}{x^2+\varepsilon^2}, c\right) = P\int_{-1}^{1} \frac{g(x)}{(x^2+\varepsilon^2)(x-c)}\,dx \tag{23}$$

where $-1 < c < 1$. While making the program formulation implementing a rule for the numerical evaluation of the CPV real integral given by equn. (23), we shall take into account the equns. (20) to (22) stated above.