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Application of Lagrange's Multiplier in Some Optimisation Problems Related to Physics

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Abstract: Several optimisation problems including those in Physics are innovated and are solved by applying Lagrange's Multipliers. Thus the minimum equivalent resistance in some electric circuits is computed subject to chosen constraints. Optimum distribution of dimensions of some water tanks with given metallic sheets for construction of the tanks ,and the corresponding maximum volume of water that can fill the tanks are determined. Finally, the minimum time of travel by a train between two terminating stations along with the corresponding optimum spacings of the intermediate stations is evaluated. Also evaluated in two problems each, the maximum value of a function of three variables restricted by a set of two constraints.

Introduction:

In textbooks of Calculus, Ref 1,3,4,5,6, many problems on "Lagrange's Multipliers" are available with/without their solutions. As for an example:A rectangular parallelopiped of some dimensions is inscribed in an ellipsoid of equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Find the maximum volume of the parallelopiped that can be inscribed in the ellipsoid.

Solution to this problem :

Let us consider a point P(x,y,z) on the ellipsoid. Then in view of the fact that the parallelopiped will be symmetrical about the centre of the ellipsoid, volume of the former is given by

V=2x.2y.2z=8xyz

subject to the constraint

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Involving Lagrange's Multiplier λ we are to form a function

F(x,y,z, λ) =8xyz+ λ(1-$\frac{x^2}{a^2}$ -$\frac{y^2}{b^2}$- $\frac{z^2}{c^2}$) such that

$\frac{\partial F}{\partial x}$=8yz-$\lambda\frac{2x}{a^2}$ =0

$\frac{\partial F}{\partial y}$=8xz-$\lambda\frac{2y}{b^2}$ =0

$\frac{\partial F}{\partial z}$=8xy-$\lambda\frac{2z}{c^2}$ =0

which lead to

$$\lambda = \frac{4a^2xyz}{x^2} = \frac{4b^2xyz}{y^2} = \frac{4c^2xyz}{z^2} = \frac{12xyz}{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}} = 12xyz$$

(by rule of ratio and proportion and by use of the constraint equation)

Eliminating λ by applying above 4 equations, we get

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}$$

Thus optimum values of the dimensions are given by

$$x_{opt}=\frac{a}{\sqrt{3}}, \quad y_{opt}=\frac{b}{\sqrt{3}}, \quad z_{opt}=\frac{c}{\sqrt{3}}$$

Hence the maximum volume of the parallelopiped accommodated in the ellipsoid is obtained as

$$V_{Max}=\frac{8}{3\sqrt{3}}abc$$

Four problems of optimisation that have not yet found home in the literature are set forth and solved by painstakingly applying Lagrange's Multiplier. Present author[1,4] innovated an optimisation problem wherein a train is depicted as undertaking a journey between two terminal stations including its travel with a uniform velocity for some time in between two consecutive intermediate stations and also including holtage time at each intermediate station. He obtained the minimum time of this journey optimising the required distance between two successive intermediate stations.

Problem no1. A circuit is prepared with coils of resistances $R_1, R_2 \; , R_3, \dots R_n,$ $respectively$ connected to cells of internal resistances $r_1, r_2 \; , r_3, \dots r_n,$

When they are connected in series, their equivalent resistance is given by $R = R_1 + R_2 \; + R_3 + \dots + R_n + r_1 + r_2 \; + r_3 \dots + r_n$ (1))The task is to optimise, ie, to minimise the equivalent resistance with respect to the external resistances subject to the constraint (1) when all external resistances R_i (i=1,2,3,....n) together with the corresponding resistances of the cells are connected in parallel. In this case the equivalent resistance R' is given by

$$\frac{1}{R'} = \frac{1}{R_1+r_1} + \frac{1}{R_2+r_2} + \frac{1}{R_3+r_3} + \quad \dots \quad + \frac{1}{R_n+r_n} \tag{2}$$

Let us choose λ as the Lagrange's Multiplier in this optimisation problem such that

$$\text{F}(R_1, R_2, R_3 \dots R_n, \lambda) = \frac{1}{R_1+r_1} + \frac{1}{R_2+r_2} + \frac{1}{R_3+r_3} + \; \dots \; + \frac{1}{R_n+r_n} - \frac{1}{\lambda}\{R - (R_1 + R_2 \; + \dots + R_n + r_1 + r_2 \; + \dots r_n)\} \tag{3}$$

$$\frac{\partial F}{\partial R_i} = \frac{-1}{(R_i + r_i \;)^2} + \frac{1}{\lambda} = 0$$

Or, $(R_i + r_i \;)^2 = \lambda$ (i =1,2,3,......n) (4)

using which in (1) is obtained

$$\sqrt{\lambda} = \frac{R}{n} \tag{5}$$

Eliminatinging λ between (5) and (4) is obtained the optimum value of each resistance in parallel connection

$$(R_i)\text{optm} = \frac{R}{n} - r_i \qquad \text{(i=1,2,3, ……..n)} \tag{6}$$

Applying (6) in (2), we get the minimum value of the equivalent resistance in parallel connection

$$R'(min) = \frac{R}{n^2} \tag{7}$$

Case2 . Let the resistor coils and the set of cells connected in series, be connected in parallel so that the equivalent resistance R''thereof turns out to be

$$\frac{1}{R''} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \dots + \frac{1}{R_n} + \frac{1}{r_1+r_2 \; + r_3 \dots r_n} \tag{8}$$

Hence the relevant function containing Lagrange's multiplier λ is herein given by

$$F=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\ \dots\ +\frac{1}{R_n}+\frac{1}{r_1+r_2\ +r_3.....r_n}\ -$$

$$\frac{1}{\lambda}\{R-(R_1+R_2\ +....+R_n+r_1+r_2\ +...r_n)\} \quad (9)$$

coupled with the same constraint equation (1) as in the previous case . Then

$$\frac{\partial F}{\partial R_i}=\frac{-1}{{R_i}^2}+\frac{1}{\lambda}=0$$

Or,$\sqrt{\lambda}$=R_i (i=1,2,3 ……n) (10)

Using (10) in (1) one gets

R=n$\sqrt{\lambda}$ +$r_1+r_2\ +...r_n$

Or, $\sqrt{\lambda}=\frac{R-\sum_{i=1}^{n} r_i}{n}=R_i$ (optimum) (11)

which gives the optimum values of resistances leading to the minimum value $R^{"}$of the equivalent resistance :

$$\frac{1}{R(min)^{"}}=\frac{n^2}{R-\sum_{i=1}^{n} r_i}+\frac{1}{\sum_{i=1}^{n} r_i} \quad (12)$$

Now let us tackle the case wherein all the external resistances and cells are connected in parallel such that the equivalent resistance $R^{"\prime}$ gives

$$\frac{1}{R^{"\prime}}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\ \dots+\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\dots\frac{1}{r_n} \quad (13)$$

As done in the previous cases

$$F=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\ \dots\ +\frac{1}{R_n}+\frac{1}{r}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_3}\dots\frac{1}{r_n}-$$

$$\frac{1}{\lambda}\{R-(R_1+R_2\ +....+R_n+r_1+r_2\ +...r_n)\} \quad (14)$$

$$\frac{\partial F}{\partial R_i}=\frac{-1}{{R_i}^2}+\frac{1}{\lambda}=0$$

Or,$\sqrt{\lambda}$=R_i (i=1,2,3 ……n) (15)

Employing (13) in (1) is obtained

R=n$\sqrt{\lambda}$+$r_1+r_2\ +r_3.....r_n$

Or,$\sqrt{\lambda}=\frac{R-\sum_{i=1}^{n} r_i}{n}$ (16)

Equating (15) to (16), one gets the optimum external resistances:

$$(R_i)\ optimum = \frac{R-\sum_{i=1}^{n} r_i}{n} \qquad \text{(i=1,2,3,...n)} \tag{17}$$

resulting in the minimum value of the equivalent resistance in parallel connection:

$$\frac{1}{R(min)''} = \frac{n^2}{R-\sum_{i=1}^{n} r_i} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \dots \frac{1}{r_n} \tag{18}$$

PROBLEM 2

This problem is different from problem1.There are n number of spots in a straight path.The distance between the first spot and the last spot is S. A sportsman starting from rest runs with a uniform acceleration f_1 and acquires a velocity after covering some distance $s_1$and immediately leaves the track.As soon as he leaves the track,a second sportsman performs the same event.Thus ith sportsman runs from rest with an acceleration f_i a distance s_i (i=1,2,3, …n) such that S=$\sum_{i=1}^{i=n} s_i$,which is the constraint equation .The optmisation problem is to find the minimum time of completion of the so called sports ,ie, the time that elapses till the last sportsman goes off the track.

SOLUTION TO THE SECOND PROBLEM

Let the ith sportsman acquire velocity v_i in time t_i just before quitting the track. Then[1]

$$s_i = \frac{1}{2} f_i {t_i}^2 \qquad v_i = f_i t_i \tag{19}$$

which yield

$$t_i = \frac{v_i}{f_i} \quad \text{and} \quad s_i = \frac{{v_i}^2}{2f_i} \tag{19.1}$$

In consequence of (19) and(19.1) the total time of completion of the sports with n participants is given by

$$T = \sum_{i=1}^{i=n} t_i = \sum_{i=1}^{i=n} \frac{v_i}{f_i} = \sum_{i=1}^{i=n} \sqrt{\frac{2s_i}{f_i}} \tag{19.2}$$

subject to the constraint

$$S = \sum_{i=1}^{i=n} s_i = \sum_{i=1}^{i=n} \frac{{v_i}^2}{2f_i} = \sum_{i=1}^{i=n} \frac{f_i {t_i}^2}{2} \tag{20}$$

$With\ \lambda$ as the Lagrange's Multiplier we can write using second parts of equations (19.2) and (20) :

$$F(v_i,\lambda)=\sum_{i=1}^{i=n} \frac{v_i}{f_i} - \lambda\{S-(\sum_{i=1}^{i=n} \frac{{v_i}^2}{2f_i})\} \quad (21)$$

$$\frac{\partial F}{\partial v_i} = -\frac{1}{f_i} + \lambda\frac{v_i}{f_i} = 0$$

Or, $v_i = \lambda$ (i=1,2,3,.......n) (22)

Substituting (22) into equation (19.1) we get

$$s_i = \frac{\lambda^2}{2f_i} \quad (23)$$

Substituting (23) in the constraint equation(20) is obtained

$$S=\sum_{i=1}^{i=n} \frac{\lambda^2}{2f_i} \quad (24)$$

Eliminating λ^2 between (23) and (24) is obtained

$$\frac{s_i}{S} = \frac{\frac{\lambda^2}{2f_i}}{\sum_{i=1}^{i=n} \frac{\lambda^2}{2f_i}}$$

$$Or,\ (s_i)optium = \frac{S}{f_i \sum_{i=1}^{i=n} \frac{1}{f_i}} \quad (25)$$

Putting (25) in (19),one gets the minimum time of travel ie minimum time of completion of the sports:

$$T= \sum_{i=1}^{i=n} \sqrt{\frac{2S}{{f_i}^2 \sum_{i=1}^{i=n} \frac{1}{f_i}}} \quad (26)$$

where the optimum time of traveling ith distance s_i is given by

$$(t_i)_{opt} = \sqrt{\frac{2S}{{f_i}^2 \sum_{i=1}^{i=n} \frac{1}{f_i}}} \quad (27)$$

and because of (19) and (27) optimum velocity of the ith participant is obtained as

$$(v_i)_{opt} = \sqrt{\frac{2S}{\sum_{i=1}^{i=n} \frac{1}{f_i}}} \quad (28)$$

This problem can be solved in another easier method.So we can rewrite

T=$\sum_{i=1}^{i=n} t_i$ and s=$\sum_{i=1}^{i=n} \frac{f_i t_i^2}{2}$

F=$\sum_{i=1}^{i=n} t_i$ -(s-$\frac{1}{\lambda}\sum_{i=1}^{i=n} \frac{f_i t_i^2}{2}$)

$\frac{\partial F}{\partial t_i}$= 1-$\frac{1}{\lambda} f_i t_i = 0$ (i=1,2,3…..n)

Or, λ=$f_i t_i$

Or,t_i=$\frac{\lambda}{f_i}$

Using which in the above constraint equation,

$$\lambda=\sqrt{\frac{2S}{\sum_{i=1}^{i=n} \frac{1}{f_i}}}$$

Using the value of λ in terms of t_i from the preceding equation, the optimum time of travel by the ith participant (to accomplish the overall minimum time of travel) is given by

$$(t_i)\text{opt}= \frac{1}{f_i}\sqrt{\frac{2S}{\sum_{i=1}^{i=n} \frac{1}{f_i}}}$$

which is identical with (27).The rest of the treatise follows as earlier.

Problem 3. The number of water tanks of rectangular shape each constructed is n with metallic sheets of 2S square units. Dimensions of the ith tank are x_i,y_i,z_i (i=1,2,3….n) respectively. The total volume of the tanks is given by

$$v=\sum_{i=1}^{n} x_i y_i z_i \tag{29}$$

In other words, n water tanks with surface area 2S, given by the equation

$$S=\sum_{i=1}^{n} (x_i y_i + y_i z_i + z_i x_i) \tag{29.1}$$

is capable of filling v cubic units of water given by (29).

Here is obtained the maximum volume of water(= volume of the tanks) with respect to the dimensions of the tanks and consequently optimum values of their dimensions subject to the constraint (29.1).

Employing Lagrange's Multiplier λ is formed the relevant function F by use of (29) and (29.1):

$F(x_i, y_i, z_i, \lambda)=\sum_{i=1}^{n} \quad x_i y_i z_i+\frac{1}{\lambda}\{S-\sum_{i=1}^{n} \quad (x_i y_i+y_i z_i + z_i x_i)\}$ (30)

$\frac{\partial F}{\partial x_i} = 0= y_i z_i -\frac{1}{\lambda}(y_i+z_i)$ (i=1,2,3 …..n)

Or, $\lambda=\frac{y_i+z_i}{y_i z_i}$ (31)

Similarly,

$\frac{\partial F}{\partial y_i} = 0= x_i z_i -\frac{1}{\lambda}(x_i+z_i)$ (i=1,2,3 …..n) (32)

$\frac{\partial F}{\partial x_i} = 0= y_i z_i -\frac{1}{\lambda}(y_i+z_i)$ (i=1,2,3 …..n) (33)

Combining (31),(32) and (33), one gets

$\lambda=\frac{1}{y_i}+\frac{1}{z_i} =\frac{1}{z_i}+\frac{1}{x_i}=\frac{1}{x_i}+\frac{1}{y_i}$ (34)

which lead to

$x_i = y_i = z_i = \frac{2}{\lambda}$ (35)

Using (35) in the constraint equation(29.1) is obtained

$S=\frac{12n}{\lambda^2}$

Or, $\lambda^2 =\frac{12n}{S}$ (36)

And also the optimum dimensions of the tanks become $x_i = y_i = z_i = \sqrt{\frac{S}{3n}}$

(i=1,2,3…..n) (37)

which ratify that all the tanks will be of cubic shape to accommodate maximum volume of water given by using (37) in (29):

$V_{max}=\frac{S}{3n}\sqrt{\frac{S}{3n}}$ (38)

with given area 2S of their surfaces.

Hence water tanks in a society/colony must be of cubical shape(with equal length, breadth and height) to ensure supply of maximum water to fill all the tanks).

Problem 4

Two terminal railway stations,viz,the first and last, ie, nth are S distance apart; thus the distance between the first and second stations is $s_1$and as such the distance between ith and (i+1) th stations is s_i.A train travels between the two terminating stations from rest to rest with uniform accelerationf_i and with uniform deceleration f'_i halting at (n-2) intermediate stations for different duration of times.If $t_1$is the time of travel between the first and the second stations,its halting time in the second station is $\epsilon t_1$where ϵ is a fraction ie <1.

Then the time spent by the train from the instant of its leaving the ith station to the instant of its leaving (i+1)th station is (1+ϵ)t_i ; (i=1,2,3,....n).

Hence the total time T of travel by the train is given by

$$T=\sum_{i=1}^{n-1} (1+\epsilon)t_i+t_n \tag{39}$$

In the last station ,ie ,nth station there is no haltage in view of the present context. This paper is aimed at determining the minimum total time (39) of travel by thee train ,ie, to minimise the function (39) subject to the constraint:

$$S=\sum_{i=1}^{n} s_i \tag{40}$$

where t_i is the time taken by the train to cover the distance s_i and v_i the maximum velocity attained due to accelerated motion during this travel.Since the train does not travel with any uniform velocity for any duration,equation (19.2) can be modified as

$$T=\sum_{i=1}^{n-1} (1+\epsilon)V_i\left(\frac{1}{f_i}+\frac{1}{f'_i}\right)+V_n\left(\frac{1}{f_n}+\frac{1}{f'_n}\right) \tag{41}$$

$$S=\sum_{1}^{n} \frac{v_i^2}{2}\left(\frac{1}{f_i}+\frac{1}{f'_i}\right) \tag{42}$$

In order to apply Lagrange's Multiplier λ for this optimisation problem as stated above, we introduce recalling (41) and (42)

$$F(V_i,f_i,f'_i\ ,\lambda)=\sum_{i=1}^{n-1} (1+\epsilon)V_i\left(\frac{1}{f_i}+\frac{1}{f'_i}\right)+V_n\left(\frac{1}{f_n}+\frac{1}{f'_n}\right)+\frac{1}{\lambda}[S-\sum_{i=1}^{n} \{\frac{v_i^2}{2}\left(\frac{1}{f_n}+\frac{1}{f'_i}\right)\}]$$

$$\frac{\delta F}{\delta V_i}=(1+\epsilon)\left(\frac{1}{f_i}+\frac{1}{f'_i}\right)-\frac{1}{\lambda}V_i\left(\frac{1}{f_i}+\frac{1}{f'_i}\right)=0$$

$$\text{Or,}V_i=\lambda(1+\epsilon) \qquad (i=1,2,3,\quad (n-1)) \tag{43}$$

$\frac{\delta F}{\delta V_n}$=$(\frac{1}{f_i} + \frac{1}{f'_i}) - \frac{1}{\lambda} V_n(\frac{1}{f_n} + \frac{1}{f'_n})$=0

Or(,V_n)=λ (44)

Substituting (43) and (44) into (42), one gets

S=$\sum_1^{n-1} \frac{\lambda^2}{2} (1+\epsilon)^2 (\frac{1}{f_i} + \frac{1}{f'_i}) + \frac{\lambda^2}{2} (\frac{1}{f_n} + \frac{1}{f'_n})$ (45)

Or, $\lambda^2 = \frac{2S}{\sum_{i=1}^{n-1} (1+\epsilon)^2 (\frac{1}{f_i}+\frac{1}{f'_i})+(\frac{1}{f_n}+\frac{1}{f'_n})}$ (46)

Using (46) in (43) and (44) are obtained optimum velocities:

$(v_i)opt = \frac{(1+\epsilon) \sqrt{2S}}{\sqrt{\ } \sum_{i=1}^{n-1} (1+\epsilon)^2 (\frac{1}{f_i}+\frac{1}{f'_i})+(\frac{1}{f_n}+\frac{1}{f'_n})}$

i=1,2,3,.......(n-1) (47)

$(v_n)opt = \frac{\sqrt{2S}}{\sqrt{\ } \sum_{i=1}^{n-1} (1+\epsilon)^2 (\frac{1}{f_i}+\frac{1}{f'_i})+(\frac{1}{f_n}+\frac{1}{f'_n})}$ (48)

Employing (47) and (48) in (41) is obtained the minimum time of travel between the two terminal stations(including the holtage time at the intermediate station) :

$$T=\sum_{i=1}^{n-1} \{\frac{(1+\epsilon)^2 \sqrt{2S}}{\sqrt{\ } \sum_{i=1}^{n-1} (1+\epsilon)^2 (\frac{1}{f_i}+\frac{1}{f'_i})+(\frac{1}{f_n}+\frac{1}{f'_n})} (\frac{1}{f_i} + \frac{1}{f'_i})\} + \frac{\sqrt{2S}}{\sqrt{\ } \sum_{i=1}^{n-1} (1+\epsilon)^2 (\frac{1}{f_i}+\frac{1}{f'_i})+(\frac{1}{f_n}+\frac{1}{f'_n})} (\frac{1}{f_n} + \frac{1}{f'_n}) \quad (49)$$

Hence the optimum distance between the ith and (i+1)th distance is given by (i=1,2,3,.....n-2)

$(s_i)opt = \frac{S(1+\epsilon)^2}{\sum_{i=1}^{n-1} (1+\epsilon)^2 (\frac{1}{f_i}+\frac{1}{f'_i})+(\frac{1}{f_n}+\frac{1}{f'_n})} (\frac{1}{f_i} + \frac{1}{f'_i})$ (50)

$(s_n)opt = \frac{S}{\sqrt{\ } \sum_{i=1}^{n-1} (1+\epsilon)^2 (\frac{1}{f_i}+\frac{1}{f'_i})+(\frac{1}{f_n}+\frac{1}{f'_n})} (\frac{1}{f_n} + \frac{1}{f'_n})$

With the corresponding times of travel between two successive intermediate stations are given by

$(t_i)opt = \frac{(1+\epsilon)^2 \sqrt{2S}}{\sqrt{\ } \sum_{i=1}^{n-1} (1+\epsilon)^2 (\frac{1}{f_i}+\frac{1}{f'_i})+(\frac{1}{f_n}+\frac{1}{f'_n})} (\frac{1}{f_i} + \frac{1}{f'_i})$ (51)

$$(t_n)opt \quad = \frac{\sqrt{2S}}{\sqrt{\sum_{i=1}^{n-1} \quad (1+\epsilon)^2 \left(\frac{1}{f_i}+\frac{1}{f'_i}\right)+\left(\frac{1}{f_n}+\frac{1}{f'_n}\right)}}\left(\frac{1}{f_n}+\frac{1}{f'_n}\right) \qquad (52)$$

Problem 4

In this problem is dealt with application of Lagrange's Multiplier with two constraints. We are to maximise a function

$$V= x^2 + y^2 + z^2 \qquad (53)$$

subject to the constraints with constants A and B:

$$A/2=a_1x+a_2y + a_3z \qquad (54)$$

$$B/2=b_1x+b_2y + b_3z \qquad (55)$$

Taking λ *and* μ *as the Lagrange's Multipliers is* framed the working function

$$F(x,y,z,\lambda) = V+\lambda\{A-(a_1x+a_2y + a_3z)\}+\mu\{B-(b_1x+b_2y + b_3z)\} \qquad (56)$$

$$\frac{\delta F}{\delta x}=2x-\lambda a_1-\mu b_1=0 \qquad (58)$$

$$\frac{\delta F}{\delta y}=2y-\lambda a_2-\mu b_2=0 \qquad (59)$$

$$\frac{\delta F}{\delta z}=2z-\lambda a_3-\mu b_3=0 \qquad (60)$$

which lead to

$$x=(\lambda a_1+\mu b_1)/2 \qquad (61)$$

$$y=(\lambda a_2+\mu b_2)/2 \qquad (62)$$

$$z=(\lambda a_3+\mu b_3)/2 \qquad (63)$$

Combining which with (54) and (55) are obtained

$$\lambda A_0 \; +\mu C =A \qquad (64)$$

$$\lambda C+\mu B_0 =B \qquad (65)$$

where

$$A_0=a_1^2+a_2^2+a_3^2 \quad B_0 \; =b_1^2+b_2^2+b_3^2 \qquad (66)$$

$$C=a_1 \, b_1 +a_2 \, b_2 \; + a_3 \, b_3 \qquad (67)$$

Solving (56) and (57) for λ and μ we have

$$\lambda = \frac{AB_0 \quad -BC}{A_0 \quad B_0 \quad - \quad C^2} \tag{68}$$

$$\mu = \frac{A_0B \quad -AC}{A_0 \quad B_0 \quad - \quad C^2} \tag{69}$$

Putting (68) and (69) in(61),(62) and(63)respectively are obtained the optimum values of x, y and z:

$$x_{opt} = a_1 \frac{AB_0 \quad -BC}{A_0 \quad B_0 \quad - \quad C^2} + b_1 \frac{A_0 \quad B \quad -AC}{A_0 \quad B_0 \quad - \quad C^2} \tag{70}$$

$$y_{opt} = a_2 \frac{AB_0 \quad -BC}{A_0 \quad B_0 \quad - \quad C^2} + b_2 \frac{A_0 \quad B \quad -AC}{A_0 \quad B_0 \quad - \quad C^2} \tag{71}$$

$$z_{opt} = a_3 \frac{AB_0 \quad -BC}{A_0 \quad B_0 \quad - \quad C^2} + b_3 \frac{A_0 \quad B \quad -AC}{A_0 \quad B_0 \quad - \quad C^2} \tag{72}$$

Therefore, the maximum value of the function V is derived by use of (70),(71) and (72) in (53):

$$V_{max} = (a_1 \frac{AB_0 \quad -BC}{A_0 \quad B_0 \quad - \quad C^2} + b_1 \frac{A_0 \quad B \quad -AC}{A_0 \quad B_0 \quad - \quad C^2})^2$$

$$+(a_2 \frac{AB_0 \quad -BC}{A_0 \quad B_0 \quad - \quad C^2} + b_2 \frac{A_0 \quad B \quad -AC}{A_0 \quad B_0 \quad - \quad C^2})^2$$

$$+(a_3 \frac{AB_0 \quad -BC}{A_0 \quad B_0 \quad - \quad C^2} + b_3 \frac{A_0 \quad B \quad -AC}{A_0 \quad B_0 \quad - \quad C^2})^2 \tag{73}$$

Now let us venture to maximise another function

R(x,y,z)=xy+yz+zx (74)

Subject to the constraint*s*

C/2=c_1x+c_2y+c_3z (75)

D/2=d_1x+d_2y+d_3z (76)

With Lagrange's Multipliers λ *and* μ the working function is written using(74)(75)and(76) :

F(X,Y,Z,λ, μ) =xz+zy+xy+$\lambda(C/2 - c_1x - c_2y - c_3z)$
$+\mu(D/2 - d_1x - d_2y \quad - d_3z\,)$

$\frac{\delta F}{\delta x} = y + z\text{-}\lambda c_1 - \mu d_1 = 0$ (77)

$\frac{\delta F}{\delta y} = z + x\text{-}\lambda c_2 - \mu d_2 = 0$ (78)

$\frac{\delta F}{\delta z} = x + y - \lambda c_3 - \mu d_3 = 0$ (79)

Combining these three equations we obtain

x+y+z=$\lambda(c_1 + c_2 + c_3)/2$

- $\mu(d_1 + d_2 + d_3)/2$ (80)

By use of (77),(78) and (79) in (80), we get the values of x, y,z in the process of finding the maximum value of the concerned expression:

x=$\lambda(-c_1 + c_2 + c_3)/2 + \mu(-d_1 + d_2 + d_3)/2$ (81)

y=$\lambda(c_1 - c_2 + c_3)/2 + \mu(d_1 - d_2 + d_3)/2$ (82)

z=$\lambda(c_1 + c_2 - c_3)/2$+$\mu(d_1 + d_2 - d_3)/2$ (83)

Substituting (81),(82) and (83) in (75) and (76) respectively, we get

C=$\lambda\{c_1(-c_1 + c_2 + c_3) + c_2(c_1 - c_2 + c_3) \quad +c_3(c_1 + c_2 - c_3)\} +$ $\mu\{c_1(-d_1 + d_2 + d_3)$+ $c_2(d_1 - d_2 + d_3)$+$c_3(d_1 + d_2 - d_3)\}$ (84)

D=$\lambda\{d_1(-c_1 + c_2 + c_3) + d_2(c_1 - c_2 + c_3) \quad +d_3(c_1 + c_2 - c_3)\} +$ $\mu\{d_1(-d_1 + d_2 + d_3)$+$d_2(d_1 - d_2 + d_3)$+$d_3(d_1 + d_2 - d_3)\}$

Or, C=λp+μq (85)

D=λr+μs (86)

where p=$c_1(-c_1 + c_2 + c_3) + c_2(c_1 - c_2 + c_3)$

$+c_3(c_1 - c_2 + c_3)\}$; $q = \{c_1(-d_1 + d_2 + d_3)$+$c_2(d_1 - d_2 +$
$d_3)$+$c_3(d_1 + d_2 - d_3)\}$ (87)

r=$d_1(-c_1 + c_2 + c_3) + d_2(c_1 - c_2 + c_3)$

$+d_3(c_1 + c_2 - c_3)\}$;

s=$d_1(-d_1 + d_2 + d_3) + d_2(d_1 - d_2 + d_3)$

$+d_3(d_1 + d_2 - d_3)\}$ (88)

Solving (85) and (86) for λ and μ are obtained

$\lambda = \frac{Cs-Dq}{sp-rq}$ and $\mu = \frac{Dp-Cr}{sp-rq}$ (89)

Substituting (89) in (81),(82)and(83)respectively we get optimum values of x,y,z:

$$x_{opt}=\frac{Cs-Dq}{2(sp-qr)}(-c_1+c_2-c_3)+\frac{Cr-Dp}{2(qr-sp)}(-d_1+d_2+d_3\ \ ) \quad (90)$$

$$y_{opt}=\frac{Cs-Dq}{2(sp-rq)}(c_1-c_2+c_3)+\frac{Cr-Dp}{2(qr-sp)}\ (d_1-d_2+d_3\ \ ) \quad (91)$$

$$z_{opt}=\frac{Cs-Dq}{2(sp-rq)}(c_1+c_2-c_3)+\frac{Cr-Dp}{2(qr-sp)}\ (d_1+d_2-d_3\ \ ) \quad (92)$$

Substituting (90),(91) and (92) in (74) is derived the maximum value of the function as

$$R_{max}=\{\frac{Cs-Dq}{2(sp-rq)}(-c_1+c_2+c_3)+\frac{Cr-Dp}{2(qr-sp)}(-d_1+d_2+d_3\ \ )\}\{\frac{Cs-Dq}{2(sp-rq)}(c_1-c_2+c_3)+\ \frac{Cr-Dp}{2(qr-sp)}\ (d_1-d_2+d_3\ \ )\}+\{\frac{Cs-Dq}{2(sp-rq)}(c_1\ \ -c_2+c_3)+\frac{Cr-Dp}{2(qr-sp)}(d_1-d_2+d_3\ \ )\}\{\frac{Cs-Dq}{2(sp-rq)}(c_1\ \ +c_2-c_3)+\frac{Cr-Dp}{2(qr-sp)}(d_1+d_2-d_3\ \ )\}+\{\frac{Cs-Dq}{2(sp-rq)}(c_1\ \ +c_2-c_3)+\frac{Cr-Dp}{2(qr-sp)}(d_1+d_2-d_3\ \ )\}\{\frac{Cs-Dq}{2(sp-rq)}(-c_1\ \ +c_2+c_3)+\frac{Cr-Dp}{2(qr-sp)}(-d_1+d_2+d_3\ \ )\} \quad (93)$$

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