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Amplified Kinematics of a Torpedo Launched and a Cannon Fired from a Naval Ship to Strike an Enemy Ship

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Abstract. Two naval ships, A and B are sailing with uniform velocities in a sea respectively along two straight line paths with an angle θ between them at junction O. While ship B is at a distance b from junction O, it launches a torpedo which moves with a uniform velocity in the sea water to hit the enemy ship A which is at a distance a from the junction O at the instant of launching the torpedo. The shortest distance between them is computed because torpedo may be launched at the time of having the shortest distance between them. A torpedo is launched and a cannon or powerless missile is fired at the same time to hit the target . The problem is mainly to find the minimum time taken by the torpedo to hit the enemy ship A and the time of flight of the projectile to hit the target. The minimum velocity of the projectile required to successfully hit the target is also found out.

Introduction

A torpedo is essentially a guided missile that happens to fly under water. It is a self- propelled weapon with an explosive. The present paper is based, firstly, on simple Mechanics necessitating no expertise on torpedo hydrodynamics/propulsion in sea water in as much as it is constrained to ply through water with a uniform velocity due to guided manipulation of its propulsion system and secondly on a projectile motion espoused on account of firing a non-powered missile or a cannon from the naval ship. However, the treatise encounters a bi-quadric equation to determine the relevant time of flight to strike the target. A number of projectile problems are solved in the textbook of Dynamics[3].

Problem 1 and Its Solution

Suppose ships A and B are sailing in the same direction along two different paths inclined at an angle θ at junction O. The ship B while at a distance b from the junction O fires at ship A while at a distance a from the junction, which is travelling with velocity u, a torpedo imparting a uniform velocity ω to it which strikes plying a distance S the enemy ship A in time t after its launch. Then by geometry , drawing figure 1,one gets

$$S^2 = (\mathrm{a} + \mathrm{ut})^2 + b^2 - 2(a + ut)b\cos\theta = \omega^2 \mathrm{t}^2 \quad (1)$$

Replacing t by $1/\tau$, expression for ω^2 can be rewritten as

$$\omega^2 = (a\tau + u)^2 + b^2\tau^2 - 2b(a\tau^2 + u\tau)cos\theta \quad (2)$$

Or, $(a^2 + b^2 - 2a\mathrm{b}\cos\theta)\tau^2 - 2(\mathrm{b}\cos\theta - \mathrm{a})\mathrm{u}\tau + \mathrm{u}^2 - \omega^2 = 0$ (3)

which is a quadratic equation with two roots which are obtained as

$$\tau = \frac{(\mathrm{b}\cos\theta - \mathrm{a})u \pm \sqrt{(\mathrm{b}\cos\theta - \mathrm{a})^2\mathrm{u}^2 - (a^2 + b^2 - 2a\mathrm{b}\cos\theta)(\mathrm{u}^2 - \omega^2)}}{a^2 + b^2 - 2a\mathrm{b}\cos\theta} \quad (4)$$

Case1.With u> ω, $(\mathrm{b}\cos\theta - \mathrm{a}) > 0$ (5)

Then (4) and (5) suggest two values of τ, ie, two different times of hitting the moving target by the torpedo obviously moving in two different paths.

$$\tau = \frac{1}{\mathrm{t}} = \frac{(\mathrm{b}\cos\theta - \mathrm{a})u \pm \sqrt{(\mathrm{b}\cos\theta - \mathrm{a})^2\mathrm{u}^2 - (a^2 + b^2 - 2a\mathrm{b}\cos\theta)(\mathrm{u}^2 - \omega^2)}}{a^2 + b^2 - 2a\mathrm{b}\cos\theta} > 0 \quad (6)$$

Hence a specific minimum time of hitting the target is given by

$$\mathrm{t}_{\min} = \frac{\mathrm{C}^2}{(\mathrm{b}\cos\theta - \mathrm{a})\mathrm{u}} \quad (7)$$

when the expression under the radical sign in (6) vanishes.

where $\mathrm{C}^2 = a^2 + \mathrm{b}^2 - 2a\mathrm{b}\cos\theta$ (8)

C is the distance between the ships at the time of launching the torpedo and the corresponding value of ω is obtained from (6) as

$$(\mathrm{b}\cos\theta - \mathrm{a})^2\mathrm{u}^2 - (a^2 + b^2 - 2a\mathrm{b}\cos\theta)(\mathrm{u}^2 - \omega^2) = 0$$

$$(\mathrm{c}^2 - \mathrm{b}^2\sin^2\theta)\mathrm{u}^2 - \mathrm{C}^2(\mathrm{u}^2 - \omega^2) = 0$$

Or, $\omega = \frac{ub\sin\theta}{C}$ (9)

If u> ω and $\mathrm{b}\cos\theta - \mathrm{a} < 0$, (10)

the time of hitting the target is hypothetically due to solution to equation (3) and as such is given by

$$\tau = \frac{1}{t} = \frac{-u(a-b\cos\theta) \pm \sqrt{(a-b\cos\theta)^2u^2-(a^2+b^2-2ab\cos\theta)(u^2-\omega^2)}}{a^2+b^2-2ab\cos\theta} < 0 \tag{11}$$

which suggests that the torpedo will not be able to hit the target.

Case2.With u< ω and $(b\cos\theta - a) > 0$ (12)

The time of hitting the target is

$$\tau = \frac{1}{t} = \frac{(b\cos\theta-a)u + \sqrt{(b\cos\theta-a)^2u^2-(a^2+b^2-2ab\cos\theta)(u^2-\omega^2)}}{a^2+b^2-2ab\cos\theta} \tag{13}$$

which suggests greater the torpedo velocity, less is the time of hitting the target.

Now if $with\ u < \omega$ and $(b\cos\theta - a) < 0,$ (14)

the time of striking the target is given by (13) indicating greater the velocity of the torpedo, less is the time of hitting the target.

Problem 2 and Its Solution

If the enemy ship A is coming from the opposite direction ie towards the junction O having the other aspects remaining the same, the distance S between the two ships when former is hit by the torpedo in time t, is given by

$$S^2 = (a - ut)^2 + b^2 - 2(a - ut)b\cos\theta = \omega^2 t^2 \tag{15}$$

whose solution is obtained replacing u by –u in the previous relevant equation

$$\tau = \frac{1}{t} = \frac{(a-b\cos\theta)u \pm \sqrt{(a-b\cos\theta)^2u^2-(a^2+b^2-2ab\cos\theta)(u^2-\omega^2)}}{a^2+b^2-2ab\cos\theta} \tag{16}$$

Case1.u> ω,
$a - b\cos\theta > 0$, as done earlier by use of (16) two times of hitting the target

$$t = \frac{C^2}{(a-b\cos\theta)u \pm \sqrt{(a-b\cos\theta)^2u^2-(a^2+b^2-2ab\cos\theta)(u^2-\omega^2)}} \tag{17}$$

Hence the specific minimum time of hitting the target is given by

$$t_{min} = \frac{C^2}{(a-b\cos\theta)u} \tag{18}$$

With velocity of the torpedo equating the expression under radical sign to zero:

$$\omega_{opt} = \frac{ub\sin\theta}{C} \qquad \text{(vide (8))} \tag{19}$$

Case 2.u> ω, $a - b\cos\theta < 0$

$$\tau = \frac{1}{t} = \frac{(a-b\cos\theta)u + \sqrt{(a-b\cos\theta)^2u^2-(a^2+b^2-2ab\cos\theta)(u^2-\omega^2)}}{a^2+b^2-2ab\cos\theta} < 0$$

which suggests that the torpedo is unable to hit the target.

Case3. u< ω, $a - b\cos\theta < 0$ implies that

greater is the velocity of the torpedo , greater is the time of hitting the target.

Geometrical Representation

Angle α of inclination of the torpedo path with respect to the path of the torpedo-launching ship in all above cases in triangle OBD is given by

Sin $\alpha = \frac{\left(\frac{a}{t} - u\right)\sin\theta}{\omega}$ (vide figure 2)

where time t of hitting the target is obtained from the above concerned equations. As for an example such an angle of launch of the torpedo can be determined by utilization of equations (7) to (9) modified by replacing u by -u:

$$\text{Sin}\,\alpha_{opt} = \frac{\left(\frac{a}{t_{min}} - u\right)\sin\theta}{\omega_{opt}} = u\left(a\,\frac{a - b\cos\theta}{C^2} - 1\right)\frac{\sin\theta}{\omega_{opt}} \quad \text{[By use of (18) and (19]}$$

$$\text{Or,} \quad \text{Sin}\,\alpha_{opt} = u\left(\frac{ab\cos\theta - b^2}{C^2}\right)\frac{\sin\theta}{\omega_{opt}} = \frac{a\cos\theta - b}{C} \qquad (20)$$

With reference to problem1 and its solution as above, equation (7) in this context is modified as

$$\frac{ut_{min}}{C} = \frac{C}{a - b\cos\theta} \qquad (21)$$

Attention is drawn to the figure1 wherein OA=a,OB=b,AB=C, <AOB=θ, AD = ut, BD = ωt , when both the ships are sailing in the same direction ; BE is drawn perpendicular to OD so that AE=$(b\cos\theta - a)$ (vide problem1)

Combining (7) and (9)

$$\frac{ut}{C} = \frac{\omega t}{b\sin\theta} = \frac{C}{b\cos\theta - a} \qquad (22)$$

$$\text{Or,} \ \frac{AD}{AB} = \frac{BD}{BE} = \frac{AB}{AE}$$

In triangles ABD and ABE, angle BAE =angle BAD and as above $\frac{BD}{AB} = \frac{BE}{AE}$

which confirms that these two triangles are similar with angle AEB=angle ABD = right angle . In order to hit the enemy ship in minimum time, the torpedo is to travel in the sea along BD perpendicular to AB ie at right angles to the initial line joining the two ships A and B whereas the distance described by the torpedo on hitting the target in minimum time because of (7) and (9 is

$$D= \omega t=\frac{bC\sin\theta}{\mathrm{b}\cos\theta-\mathrm{a}} \qquad (23)$$

With reference to problem2 , solution to equation (18) can be rewritten as

$$\frac{ut_{min}}{C} = \frac{C}{\mathrm{a}-\mathrm{b}\cos\theta} \qquad (24)$$

Attention is drawn to the figure2 wherein OA=a,OB=b,AB=C, <AOB=θ, $\mathrm{AD} = \mathrm{ut}, \mathrm{BD} = \omega\mathrm{t}$, when two the ships are sailing in the opposite direction ; BE is drawn perpendicular to OD so that AE=$(\mathrm{a} - \mathrm{b}\cos\theta)$. Vide problem2.

In line with (7) and (9)

$$\frac{\mathrm{ut}}{\mathrm{C}} = \frac{\omega\mathrm{t}}{\mathrm{b}\sin\theta} = \frac{C}{\mathrm{a}-\mathrm{b}\cos\theta} \qquad (25)$$

$$\frac{\mathrm{AD}}{\mathrm{AB}} = \frac{\mathrm{BD}}{\mathrm{BE}} = \frac{\mathrm{AB}}{\mathrm{AE}}$$

which ratifies that with angle AEB=angle DEB = right angle ,two right- angled triangles ABE and DAE are similar and hence by geometry angle ABD=right angle.

Hence In order to hit the enemy ship in minimum time , the torpedo is to travel in the sea at right angles to the initial line joining the two ships A and B while the distance described by the torpedo on hitting the target in minimum time because of (7) and (9) modified is given by

$$D= \omega t=\frac{bC\sin\theta}{\mathrm{a}-\mathrm{b}\cos\theta} \qquad (26)$$

In other words it can travel with minimum velocity as derived below to hit the target.

Use of Calculus to Tackle the Problems

Differentiating (2) with respect to t,

$$\frac{\mathrm{d}\omega^2}{d\tau} = 2(a\tau + \mathrm{u})\mathrm{a} + 2\mathrm{b}^2\tau - 2\mathrm{b}(2a\tau + \mathrm{u})\cos\theta = 0; \frac{d^2\omega^2}{d\tau^2} = C^2 > 0$$

which reveal that there exists a minimum velocity of travel by the torpedo to hit the target in time t:

$$\tau = \frac{1}{\mathrm{t}} = \frac{(\mathrm{b}cos\theta-a)u}{\mathrm{a}^2+\mathrm{b}^2-2\mathrm{ab}\cos\theta} \qquad (\mathrm{b}cos\theta - a) > 0 \qquad (27)$$

and in consideration of equation(19) is obtained or using (27) in (2)

$$\omega_{\mathrm{min}}=\frac{ub\sin\theta}{C} \qquad (28)$$

Hence the torpedo traveling with this minimum velocity can hit the target.

In case of the two ships travelling in the opposite directions using Calculus,

$$\tau = \frac{1}{t} = \frac{(a-b\cos\theta)u}{a^2+b^2-2ab\cos\theta} \qquad \omega_{min}=\frac{ub\sin\theta}{C} \tag{28.1}$$

where $(a - b\cos\theta) > 0$

Two Ships Sailing in Paralell Paths

Let the two naval ships sail in the same direction parallel to each other having the same parameters as earlier. This is illustrated in Fig.3, which as a trivial case suggests

$$\omega t\sin\beta = a + ut - b, \tag{29}$$

$$\omega t\cos\beta = h = \text{distance between the two parallel paths} \tag{30}$$

Or, $t=\frac{h\tan\beta-a+b}{u}$ (31)

which gives the time taken to strike the target after launch of the torpedo and the corresponding velocity ω of the torpedo is obtained by eliminating t

between (31) and (30),where h is the shortest distance between the parallel paths and β the inclination of the launch velocity with the shortest distance.

Hence $\omega = \frac{u}{\sin\beta-\frac{a-b}{h}\cos\beta} = \frac{u\cos\alpha}{\sin(\beta-\alpha)}$ (32)

where α is the inclination of the initial line joining the two ships with the above mentioned shortest distance between them.

Squaring and adding (29) and (30) is obtained supposedly the time t of hitting :

$\omega^2=\{(a-b)\tau+u\}^2+h^2\tau^2$

Or, $\{(a-b)^2+h^2\}\tau^2+2u(a-b)\tau+(u^2-\omega^2)=0$

Or, $\tau = \frac{1}{t} = \frac{-u(a-b)\pm\sqrt{u^2(a-b)^2-\{(a-b)^2+h^2\}(u^2-\omega^2\}}}{(a-b)^2+h^2}$ (33)

Case 1.u> $\omega, a > b$.

From (33) t<0 suggests that the torpedo is unable to hit the target ship.

Cause2.u< $\omega, a > b$. t>0 indicates that the torpedo is able to hit the target in time

$t=\frac{(a-b)^2+h^2}{-u(a-b)+\sqrt{u^2(a-b)^2-\{(a-b)^2+h^2\}(u^2-\omega^2\}}}$ (34)

Case3. u> $\omega, a < b$

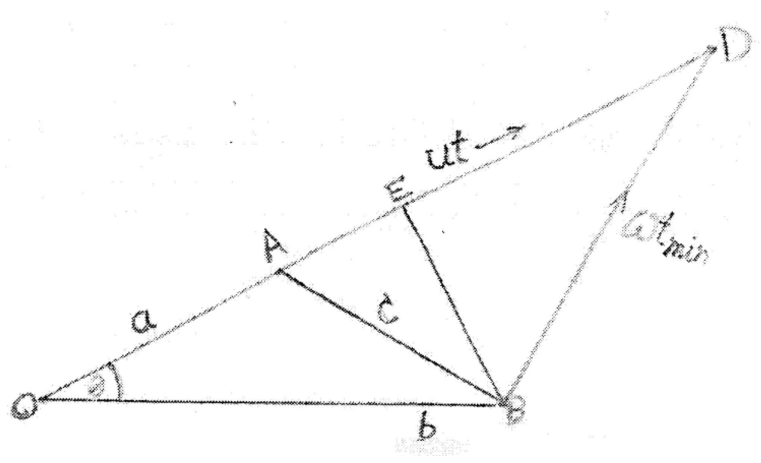

Figure 1

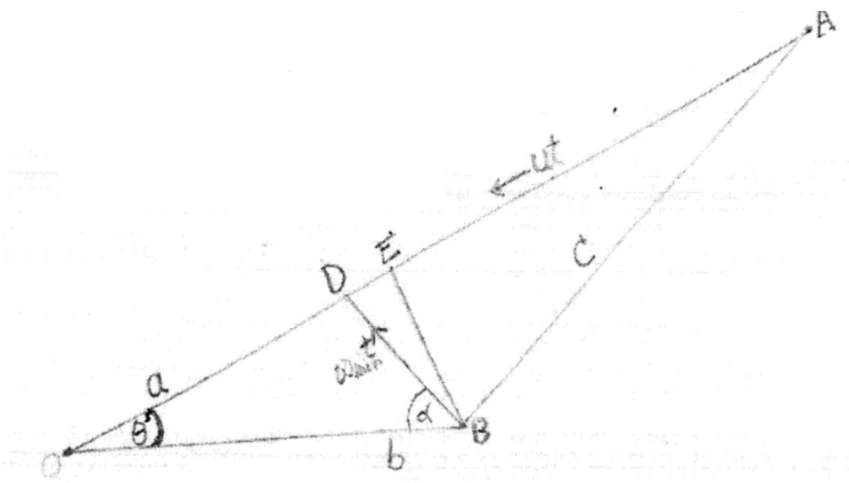

Figure 2

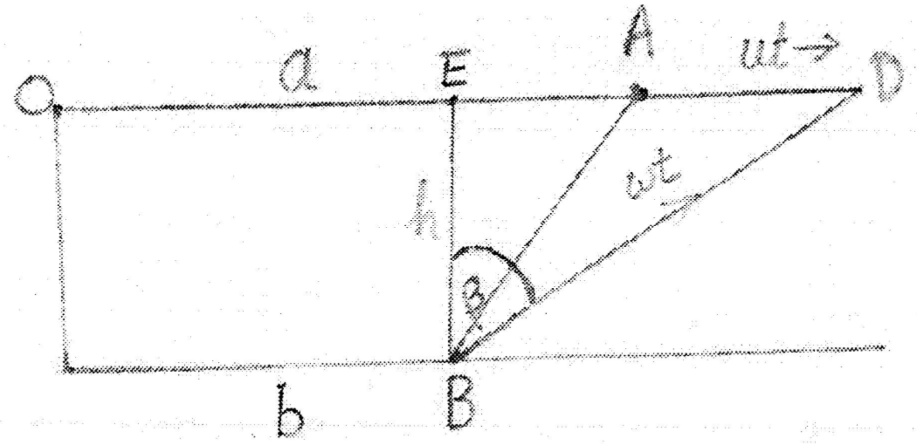

Figure 3

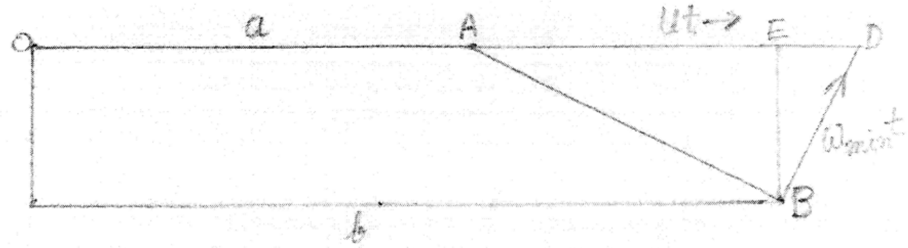

Figure 4

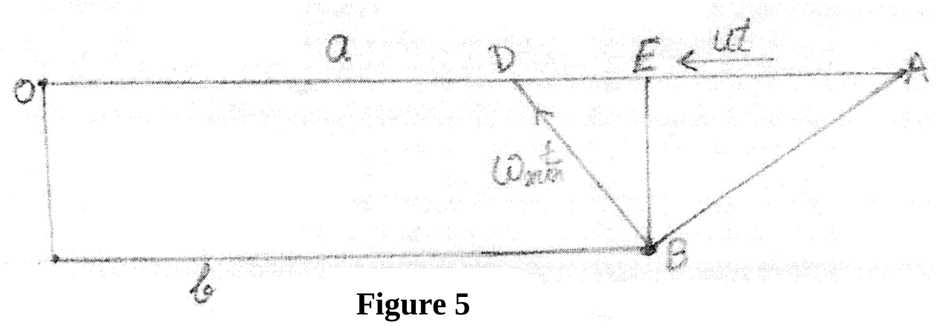

Figure 5

$$t=\frac{(a-b)^2+h^2}{-u(a-b)\pm\sqrt{u^2(a-b)^2-\{(a-b)^2+h^2\}(u^2-\omega^2\}}} \quad >0 \tag{35}$$

revealing two times of hitting the target obviously with two different values of ω. The specific minimum time of hitting the target is given by equating the expression under radical sign to zero so as to also obtain the optimum value of ω:

$$t_{min}=\frac{c^2}{u(b-a)} \qquad \omega_{opt}=\frac{hu}{c} \qquad b>a \tag{36}$$

The optimum distance travelled by the torpedo to hit the target is hence given by

$$D_{opt}=\frac{hc}{(b-a)} \tag{37}$$

where $c^2 = (a-b)^2 + h^2$ (37.1)

Relationship (36) can also be derived by Calculus. $t_{opt}=\frac{c^2}{u(b-a)}$, $\omega_{min}=\frac{hu}{c}$

By geometry OA =a, OB=b, AB=c=Initial distance between the two ships, BE=h, BD=$\frac{hc}{(b-a)}$, AE=b−a, AD=ut_{min} in Figure 4:

In triangles ABE and ABD, angle BAE=angle BAD =common and from (37)

$\frac{BD}{AB}=\frac{BE}{AE}$ which implies that these two triangles are similar with BE ⊥ AD and are right − angled . Hence angle ABD=angle AEB =right angle ie AB⊥ BD. In order to strike the target ship in minimum time with optimum velocity given by relations (36) the torpedo has to be launched and to travel at right angles to the line of sight of the target from the attacking ship.

Case 4. u< ω, $a < b$

$$t=\frac{(a-b)^2+h^2}{-u(a-b)+\sqrt{u^2(a-b)^2-\{(a-b)^2+h^2\}(u^2-\omega^2\}}} \quad >0$$

which confirms that the torpedo is able to strike the target in this time and that greater is the torpedo velocity ω less is the time of hitting the target.

If the target ship is sailing in the opposite direction u, is replaced by –u in the relevant previous equations such that

$$\omega^2=\{(a-b)\tau-u\}^2+h^2\tau^2 \qquad a>b$$

For maximum or minimum value of ω^2 *ie* ω,

$$\frac{d\omega^2}{d\tau} = 2\{(\mathrm{a}-\mathrm{b})\tau - \mathrm{u}\}(\mathrm{a}-\mathrm{b}) + 2\mathrm{h}^2\tau = 0$$

$$\frac{\mathrm{d}^2\omega^2}{d\tau^2}=2\{(\mathrm{a}-\mathrm{b})^2 + \mathrm{h}^2\} > 0 \qquad (1/\tau = \mathrm{t})$$

which confirm that there exists a minimum velocity of the torpedo to hit the target:

$$t_{opt}= \frac{\mathrm{c}^2}{\mathrm{u}(\mathrm{a}-\mathrm{b})} \qquad \omega_{min} = \frac{hu}{c} \qquad \mathrm{a}>\mathrm{b} \tag{38}$$

In this case also by geometry it can be established that the torpedo can hit the target travelling at right angles to the line of sight of the target at the time of its launch with minimum velocity in optimum time given by (38).

In both the cases of travel in the same direction and in the opposite direction there exists a same minimum velocity of the torpedo or equivalently a same minimum time of travel by the torpedo to reach the target.

Target Ship and Striking Ship are Sailing at Right Angles

Taking into consideration of the above text, if the target ship is moving towards junction O, with the same parameters as before, the distance between them is :

$$(a-ut)^2 + \mathrm{b}^2 = \omega^2 t^2 \tag{39}$$

$$\text{Or,}\frac{\mathrm{a}^2+\mathrm{b}^2}{t^2} - \frac{2au}{t} + \mathrm{u}^2 - \omega^2=0 \tag{39.1}$$

Solving the quadratic equation (31), one gets

$$\mathrm{t} = \frac{\mathrm{a}^2+\mathrm{b}^2}{au\pm\sqrt{(\mathrm{a}^2+\mathrm{b}^2)\omega^2-\mathrm{u}^2\mathrm{b}^2}} \tag{39.2}$$

which indicates that there are two times of hitting the target if u> ω .Then from (33.2) , the specfic minimum time of reaching the target is

$$t_{min} = \frac{\mathrm{a}^2+\mathrm{b}^2}{au} \tag{40}$$

And consequently substituting (34) into (33) is obtained the corresponding velocity of launching the torpedo;

$$\omega = \frac{ub}{\sqrt{\mathrm{a}^2+\mathrm{b}^2}} \tag{41}$$

Using (40) and (41), one gets the optimum distance travelled by the torpedo:

$$D_{opt}=\frac{b\sqrt{a^2+b^2}}{a} \tag{41.1}$$

which suggests that in the right angled triangles AOB and BOD AO=a , OB= b, AB=$\sqrt{a^2+b^2}$, $BD = D_{opt}$ Angle AOB=Angle BOD =Right angle, see Figure 6

$\frac{OB}{BD}=\frac{OA}{AB}$. Hence these two triangles are similar confirming that

Angle OAB=Angle OBD and Angle ODB=Angle ABO. Hence the torpedo is to travel along BD making with the path of the attacking ship an angle equal to the angle between the initial line of sight of the target and the path of the target so as hit the target in minimum time or equivalently the torpedo can travel with a minimum velocity for the same. The same results analogous to (27) can be reached by use of Calculus.

Target Ship Moving Away from the Junction and the Other Ship Towards the Junction

If the velocity of the striking ship be v, the distance S between them after time t

$$S^2 = (a+ut)^2++(b-vt)^2-2(a+ut)(b-vt)\cos\theta \tag{42}$$

To find the shortest distance S_d between them and time t_s to take place, we have

$$\frac{dS^2}{dt} = -2v(b-vt)+2u(a+ut)-2\{u(b-vt)-v(a+ut)\}cos\theta = 0$$

$$\text{Or, } t_s = \frac{v(b-a\cos\theta)-u(a-b\cos\theta)}{u^2+v^2+2uv\cos\theta} \tag{43}$$

$$S_d^2 = \left[a+u\left\{\frac{v(b-a\cos\theta)-u(a-b\cos\theta)}{u^2+v^2+2uv\cos\theta}\right\}\right]^2+\left[b-v\left\{\frac{v(b-a\cos\theta)-u(a-b\cos\theta)}{u^2+v^2+2uv\cos\theta}\right\}\right]^2$$

$$-2(a+u\frac{v(b-a\cos\theta)-u(a-b\cos\theta)}{u^2+v^2+2uv\cos\theta})(b-v\frac{v(b-a\cos\theta)-u(a-b\cos\theta}{u^2+v^2+2uv\cos\theta})\cos\theta$$

$$= [\frac{(av+bu)(v+u\cos\theta)}{u^2+v^2+2uv\cos\theta}]^2+[\{\frac{(ub+va)(u+v\cos\theta)}{u^2+v^2+2uv\cos\theta}\}]^2 -$$
$$2\frac{(av+bu)(v+u\cos\theta)}{u^2+v^2+2uv\cos\theta}\frac{(ub+va)(u+v\cos\theta)}{u^2+v^2+2uv\cos\theta}\cos\theta \tag{44}$$

$$=\frac{(av+bu)^2\{((v+u\cos\theta)^2+(u+v\cos\theta)^2\}-2\{(v+u\cos\theta)(u+v\cos\theta)cos\theta\}}{(u^2+v^2+2uv\cos\theta)^2}$$

$$=\frac{(av+bu)^2\{(u^2+v^2)(1+\cos^2\theta)-2(u^2+v^2)\cos^2\theta+2uvcos\theta-2uv\cos^3\theta\}}{(u^2+v^2+2uv\cos\theta)^2}$$

$$=\frac{(av+bu)^2\{(u^2+v^2)(\sin^2\theta)+2uvcos\theta\sin^2\theta\}}{(u^2+v^2+2uv\cos\theta)^2}$$

$=\frac{(av+bu)^2\sin^2\theta}{u^2+v^2+2uv\cos\theta}$

$$S_d = \frac{(av+bu)sin\theta}{\sqrt{(u^2+v^2+2uv\cos\theta}} \tag{45}$$

If they are travelling at right angles ,ie $\theta = 90^0$,

the shortest distance$=\frac{(av+bu)}{\sqrt{u^2+v^2}}$ in time $=\frac{vb-ua}{u^2+v^2+2uv\cos\theta}$ (46)

Torpedo Launched when the Ship and the Target are Shortest Distance Apart

By the time of occurrence of the above shortest distance, the distances of the attacking ship and the target ship from the junction O due to (44) are respectively given by

$A=u\frac{(av+bu)(v+u\cos\theta)}{u^2+v^2+2uv\cos\theta}$ and $B=v\frac{(ub+va)(u+v\cos\theta)}{u^2+v^2+2uv\cos\theta}$ (47)

Let the torpedo be launched as above, travel with uniform velocity Ω in the sea and hit the target in time T. Then in the same line as earlier,

$$(A+uT)^2 + B^2 = \Omega^2 T^2 \tag{48}$$

Further study can be carried out in totality as is done earlier.

Attacking Ship Fires Cannon to Strike the Target Ship: Projectile Motion

Let V be the velocity of projection of the projectile at an angle δ to the horizontal resulting in the time of flight t and g the acceleration due to gravity ;we have

$$(a+ut)^2 + b^2 - 2(a+ut)b\cos\theta = V^2(\cos^2\delta)t^2 \tag{49}$$

$$t=\frac{2Vsin\delta}{g} \tag{50}$$

This is illustrated in Figure 7

Eliminating δ between equations (49) and (50) is obtained

$$\frac{g^2t^4}{4}+(u^2-V^2)t^2+2u(a-b\cos\theta)t+(a^2+b^2-2ab\cos\theta)=0$$

$$\text{Or, } t^4+\frac{4(u^2-V^2)t^2}{g^2}+\frac{8u}{g^2}(a\text{-}b\cos\theta)t+\frac{4(a^2+b^2-2ab\cos\theta)}{g^2}=0 \tag{51}$$

which represents a bi-quadratic equation to be solved in a tedious and cumbersome process. To facilitate solution to equation (51) is now recalled

$$t^4 + 2pt^2 + 4qt + r = 0 \tag{52}$$

where $2p=\frac{4(u^2-V^2)}{g^2}$, $4q=\frac{8u}{g^2}(a\text{-}b\cos\theta)$, $r = \frac{4(a^2+b^2-2ab\cos\theta)}{g^2}$ (53)

Equation (52) is further reproduced as

$$(t^2 + p + e)^2 - (p + e)^2 - 2et^2 + 4qt + r = 0$$

Or, $(t^2 + p + e)^2 = (p + e)^2 + 2et^2 - 4qt - r$

Or, $(t^2 + p + e)^2 = (p + e)^2 + 2e(t^2 - \frac{4q}{2e}t) - r$

Or, $(t^2 + p + e)^2 = (p + e)^2 + 2e(t^2 - \frac{2q}{e}t + \frac{q}{e^2}^2 - \frac{q}{e^2}^2) - r$

Or, $(t^2 + p + e)^2 = (p + e)^2 + 2e\left(t - \frac{q}{e}\right)^2 - \frac{2q^2}{e} - r$ (54)

We have introduced e in the above equation to evolve a square involving t^2 on the left hand side and another square involving t on the right hand side and the remaining expression involving e which is equated to zero to obtain the value of e for such adjustment and so

$$(p + e)^2 - \frac{2q^2}{e} - r = 0 \tag{55}$$

which is a cubic equation and is written in a proper form with relevant substitution

$$e^3 + 2pe^2 + (p^2 - r)e - 2q^2 = 0$$

Let us put $e= z-\frac{2p}{3}$ (56)

in the above equation so that it reduces to the required form

$$z^3 - 3\frac{2p}{3}z^2 + 3\frac{4p^2}{9}z - \frac{8p^3}{27} + 2p(z - \frac{2p}{3})^2 + (p^2-r)(z-\frac{2p}{3}) - 2q^2 = 0$$

Or, $z^3 - \left(\frac{p^2}{3} + r\right)z + \frac{2p}{3}r - \frac{2p^3}{27} - 2q^2 = 0$ (57)

Assuming from equation (57)

$3Q=\frac{p^2}{3} + r$ and $-2P =\frac{2p}{3}r - \frac{2p^3}{27} - 2q^2$ (58)

the same reduces to the form

$$z^3 - 3Qz - 2P = 0 \tag{59}$$

which is solved in a conventional method with proper substitution :

$$z=m^{\frac{1}{3}}+n^{\frac{1}{3}} \tag{60}$$

cubing which one gets

$$z^3 -(m+n) -3m^{\frac{1}{3}}n^{\frac{1}{3}}z \quad =0$$

so that comparing this equation with (59) , is obtained

$$m+n=2P \quad \text{and} \quad mn= Q^3 \tag{61}$$

To find the values of m and n we form a quadratic equation with the help of (61) and variable x:

$$x^2-2Px+ Q^3 \quad =0 \tag{62}$$

whose roots are m and n given by

$$\text{m or n}= P\pm\sqrt{P^2-Q^3} \tag{63}$$

Using (56),(60) and (63) we get

$$e=(P\pm\sqrt{P^2-Q^3})^{\frac{1}{3}}+(P\pm\sqrt{P^2-Q^3})^{\frac{1}{3}}-\frac{2p}{3} \tag{64}$$

Now (54) and (55) lead to

$$(t^2+p+e)^2=2e\left(t-\frac{q}{e}\right)^2 \qquad e>0 \tag{65}$$

Or, $t^2+p+e=\pm\sqrt{2e}\left(t-\frac{q}{e}\right)$

$$t^2-\sqrt{2e}t+\frac{q}{e}\sqrt{2e}+p+e=0 \quad \text{and} \quad t^2+\sqrt{2e}t-\frac{q}{e}\sqrt{2e}+p+e=0 \tag{66}$$

Solving (65) we get four values of t ,out of which only real and positive values of t with given values of the constant parameters e, p, q will stand for time(s) of flight to hit the target.

$$t=\frac{\sqrt{2e}\pm\sqrt{2e-4(\frac{q}{e}\sqrt{2e}+p+e)}}{2}>0 \quad \text{and} \quad t=\frac{-\sqrt{2e}+\sqrt{2e+4(\frac{q}{e}\sqrt{2e}-p-e)}}{2}>0 \tag{67}$$

which ratifies that there are one to three times of flight to successfully hit the moving target depending on the values of constants e, p and q, for example, initial velocity, angle of projection etc. It can be visualized that to strike a stationary or moving object ,there must exit some conditions involving constant parameters to be fulfilled, viz,the expressions under the above radical signs must be positive. See Figure 7.

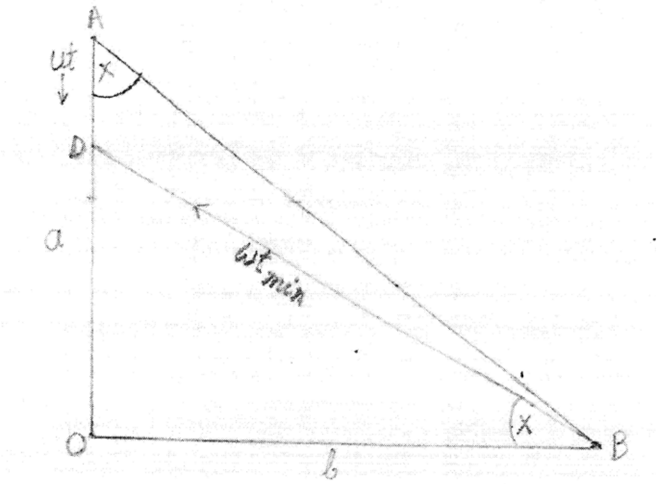

Figure 6

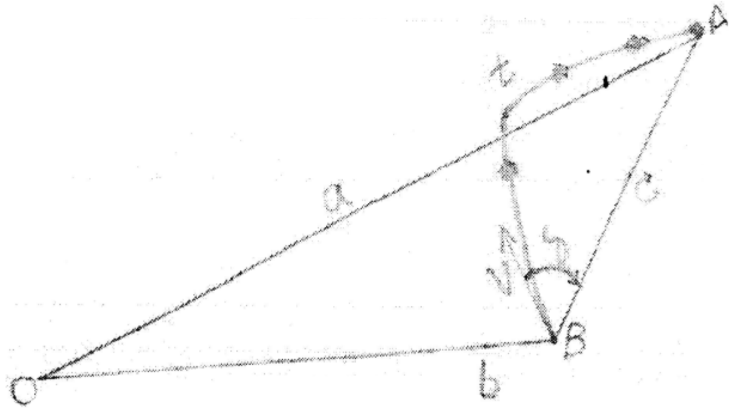

Figure 7

Minimum Velocity of the Projectile to Hit the Target

To find as above we rearrange equation (51):

$$t^4 + \frac{4(u^2 - V^2)t^2}{g^2} + \frac{8u}{g^2}(\text{a-bcos}\theta)t + \frac{4(a^2 + b^2 - 2ab\cos\theta)}{g^2} = 0$$

$$\frac{4(V^2 - u^2)t^2}{g^2} = \frac{8u}{g^2}(a - b\cos\theta)t + \frac{4(a^2 + b^2 - 2ab\cos\theta)}{g^2} + t^4$$

Or, $4V^2 = g^2t^2 + 4u^2 + \frac{8u(a - b\cos\theta)}{t} + \frac{4(a^2 + b^2 - 2ab\cos\theta)}{t^2}$ (68)

For maximum or minimum of V^2 ie V,with respect to time t, given the other constant parameters, its derivative with respect to t is equated to zero;

$$\frac{4dV^2}{dt} = 2g^2t - \frac{8u(a - b\cos\theta)}{t^2} - \frac{8(a^2 + b^2 - 2ab\cos\theta)}{t^3} = 0 \quad (69)$$

$$4\frac{d^2V^2}{dt^2} = 2g^2 + \frac{16u(a - b\cos\theta)}{t^3} + \frac{24(a^2 + b^2 - 2ab\cos\theta)}{t^4} > 0 \quad (70)$$

which reveals that there exists a minimum velocity of projection obtainable by

$$t^4 - \frac{4u(a - b\cos\theta)t}{g^2} - \frac{4(a^2 + b^2 - 2ab\cos\theta)}{g^2} = 0 \quad (71)$$

which is to be solved as under

$$t^4 + 2rt^2 + r^2 - r^2 - 2rt^2 - \frac{4u(a - b\cos\theta)t}{g^2} - \frac{4(a^2 + b^2 - 2ab\cos\theta)}{g^2} = 0$$

Or, $(t^2 + r)^2 - 2r\{t^2 + \frac{2u(a - b\cos\theta)t}{g^2 r} + (\frac{u(a - b\cos\theta)}{g^2 r})^2 - (\frac{u(a - b\cos\theta)}{g^2 r})^2\} - r^2$

$$- \frac{4(a^2 + b^2 - 2ab\cos\theta)}{g^2} = 0$$

Or, $(t^2 + r)^2 - 2r\{(t + \frac{u(a - b\cos\theta)}{g^2 r})^2 - (\frac{2u(a - b\cos\theta)}{g^2 r})^2\} - r^2$

$$- \frac{4(a^2 + b^2 - 2ab\cos\theta)}{g^2} = 0$$

$$(t^2 + r)^2 = 2r\{\left(t + \frac{u(a - b\cos\theta)}{g^2 r}\right)^2 - \left(\frac{2u(a - b\cos\theta)}{g^2 r}\right)^2\} + r^2 + \frac{4(a^2 + b^2 - 2ab\cos\theta)}{g^2}$$

We have introduced a constant r in the above equation to yield a "square" involving t on both sides leading to a term of cubic expression in r on the right hand side, which is equated to zero to evaluate the value of r:

$$(t^2 + r)^2 = 2r(t + \frac{u(a - b\cos\theta)}{g^2 r})^2 \quad (72)$$

and $2r\left(\frac{2u(a-b\cos\theta)}{g^2 r}\right)^2 - r^2 - \frac{(a^2+b^2-2ab\cos\theta)}{g^2} = 0$ (73)

Using (72) we are supposed to get four roots :

$$t^2 - \sqrt{2r}\,t + r - \sqrt{2r}\,\frac{u(a-b\cos\theta)}{g^2 r} = 0 \quad (74)$$

$$\text{Or, } t_{0pt} = \frac{\sqrt{2r} + \sqrt{4\sqrt{2r}\frac{u(a-b\cos\theta)}{g^2 r} + 2r}}{2} > 0 \quad (75)$$

and $t^2 + \sqrt{2r}\,t + r + \sqrt{2r}\,\frac{u(a-b\cos\theta)}{g^2 r} = 0 \rightarrow t < 0$ $\quad (a - b\cos\theta) > 0$

Therefore the minimum velocity of projection gives rise to the time of hitting the target given by (69).To evaluate the value of r, relationship (73) turns out to be a cubic equation that is solved:

$$\text{Or, } r^3 + \frac{(a^2+b^2-2ab\cos\theta)}{g^2} r - 2\left(\frac{2u(a-b\cos\theta)}{g^2}\right)^2 = 0 \quad (76)$$

Let $r = m^{\frac{1}{3}} + n^{\frac{1}{3}}$ (77)

cubing which one gets

$$r^3 - (m+n) - 3m^{\frac{1}{3}}n^{\frac{1}{3}}r = 0$$

so that comparing this equation with (76) ,one gets

$$m + n = 2\left(\frac{2u(a-b\cos\theta)}{g^2}\right)^2 \quad \text{and} \quad 3m^{\frac{1}{3}}n^{\frac{1}{3}} = \frac{-(a^2+b^2-2ab\cos\theta)}{g^2} \quad (78)$$

As such m,n are roots of the quadratic equation

$$x^2 - 2\left(\frac{2u(a-b\cos\theta)}{g^2}\right)^2 x - \left(\frac{(a^2+b^2-2ab\cos\theta)}{3g^2}\right)^3 = 0 \quad (79)$$

$$\text{Or, } x = \left(\frac{2u(a-b\cos\theta)}{g^2}\right)^2 \pm \sqrt{\left(\frac{2u(a-b\cos\theta)}{g^2}\right)^4 + \frac{(a^2+b^2-2ab\cos\theta)^3}{27g^6}} = m,n$$

Recalling (77) is obtained

$$r = \left\{\left(\frac{2u(a-b\cos\theta)}{g^2}\right)^2 + \sqrt{\left(\frac{2u(a-b\cos\theta)}{g^2}\right)^4 + \frac{(a^2+b^2-2ab\cos\theta)^3}{27g^6}}\right\}^{\frac{1}{3}}$$

$$+\left\{\left(\frac{2u(a-b\cos\theta)}{g^2}\right)^2 - \sqrt{\left(\frac{2u(a-b\cos\theta)}{g^2}\right)^4 + \frac{(a^2+b^2-2ab\cos\theta)^3}{27g^6}}\right\}^{\frac{1}{3}} \quad (80)$$

Substituting (80) into (75) we get the relevant time of flight t_{opt} which is again substituted in (68) to achieve minimum velocity V_{min} of projection which along with the expression for t_{opt} is ultimately substituted into (50) to obtain optimum value of the angle of projection:

$$\mathrm{Sin}\delta_{opt}=\frac{gt_{opt}}{2V_{min}} \tag{81}$$

It is worthwhile mentioning that given the numerical values of the constant parameters, the foregoing analysis though cumbersome and laborious can cater to the expertise utmost from academic point of view and can be fed into software in computer to acquire the desired result.

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