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Analysis of Projectile Motion with a Drag Force using MATLAB

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Abstract. The projectile motion, a branch science of exterior ballistics, the 61st branch of physics with drag force has been a complex physics and an interesting phenomenon of investigation for many centuries till today. It deals with the study of substantially below the speed of light. The advancement of the problem has come from three fields: physics, mathematics, and computation for various industry, civil, and military applications. As a physical and mathematical in nature, the focus has been given on the methods for computation, simulation and analysis of the governing equations of motion. In this paper, the projectile motion with various drag forces: (a) zero drag, (b) linear drag, (c) quadratic drag are computed, simulated and analysed for predicting various trajectories profiles with three set of input parameters using MATLAB source codes.

1. Introduction

> "No human investigation can claim to be scientific if it doesn't pass the test of mathematical proof."
>
> _Leonardo da Vinci

Projectile motion is one topic in an introductory physics course for undergraduate student. Any object released into the air is called a projectile. It is cast, fired, shot, launched, flung, heaved, hurled, pitched, tossed, or thrown. Simply, it is defined as "an object, projected by an applied exterior force or impulse and continuing in motion by virtue of its own inertia over a period of time”. Generally, a projectile is subjected to fired, launched, or thrown through space by the exertions of a force by a suitable means, behaves dynamically and the path followed by a projectile is called its trajectory. When an object is “launched,” what is its subsequent motion? By Newton‟s laws, one needs to know the forces acting on the object. Potentially, these include gravity, air resistance (drag), a force due to the object spinning and a force due to asymmetries in the object‟s shape. Examples of projectile motion include: (a) A baseball or a football or a basketball or a tennis ball that has pitched, batted, or thrown; (b) A bullet, or a shot or a grenade the instant it exits the barrel of a gun or rifle;(c) A rocket or a missile or a cannonball fired into space, Figure 1 ;(d) A bus driven off an uncompleted bridge;(e) A moving airplane in the air with its engines and wings disabled;(f) A runner in mid stride (since they momentarily lose contact with the ground); (g) The space shuttle or any other spacecraft after main engine cut off; (h) The motions of a slate blown off a roof and a piece of mud or small stone thrown up from the road against a car windscreen, that of the shattered glass of a windscreen; (i) The drops of water that from the jet from a hosepipe and so on. The Figure 2 shows a phase plane of projectile trajectories.

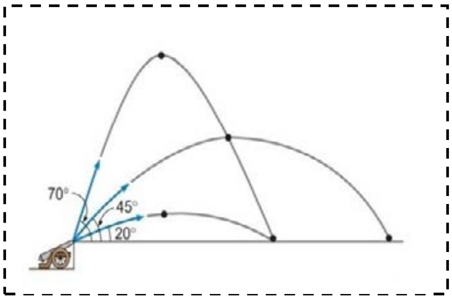

Fig. 1: Motion of a Cannonball Projectile

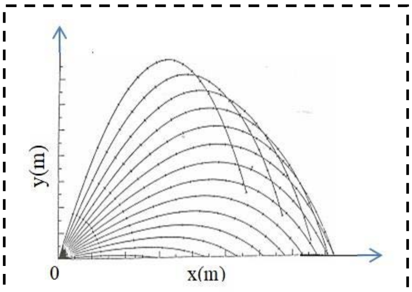

Fig. 2: Phase Plane Profiles of a Projectile Motion

The study of parabolic motion, without drag force, is a common example in introductory physics course for undergraduate student. If we negligible air resistance, a projectile follows a curved trajectory or curved path that is a parabola. Introducing drag with other effects forces into the projectile motion study gives rise to a problem that is difficult to solve analytically, except in a few particular cases. Educational studies of projectile motion under the influence of various drag forces have long been studied. Linking with its motion, a trajectory is defined as an imagined trace of positions followed by an object moving through space. The trajectory of a projectile motion is affected by gravity, aerodynamic forces and others.

The problem of a projectile moving in a resisting medium such as air is a very old one having first received serious attention in the mid to late seventeenth century. Historically the problem was of great importance since it was intimately connected with one of the most pressing problems of its day, the study of external ballistics. Fortunately nowadays more peaceful interest in projectiles moving through resisting media is to be found in applications such as those associated with various applications.

This paper attempts to describe the motion of a projectile on the assumption that the forces acting, besides the forces of gravity, are a so-called drag in the direction on the tangent to the trajectory opposite to the motion of the projectile. There are numbers of factors, which affect the motion of the projectile; some associated with the projectile itself and others with the atmosphere through which the projectile is moving.

A numerical study of projectile motion with linear and quadratic dependence on projectile speed has been studied by many authors. The drag force has been considered in a variety of conditions, showing that the trajectory may be approximated under various effects. The trajectory equations have been coded MATLAB source codes and to visualize the trajectory paths of a specific projectile with and without air drag. The aim of this paper studies the physical, mathematical and computational aspects on of projectile motion concepts for a variety of applications in recreational, forensic, sports, and military and so on.

2. History of Projectile Motion

In our physical world, motion is ubiquitous and everything in the universe is controlled by the application of two principal ideas, force and motion. The study of these ideas is called dynamics. The study of dynamics calculates and analyse the trajectory of a projectile motion. This is what science is all about - studying the environment around us and predicting the future.

The subject of projectile motion has been studied for many centuries and, drawing on fields as diverse as physics, mechanics and fluid dynamics, the subject is relatively complicated, both from the physical and mathematical point of view. It is only with an understanding of the physical principles involved, that one can fully appreciate the complexities in the mathematical formulation of the problem, and also realistically interpret the predicted trajectories. Accordingly, in the interests of the reader not familiar with projectile motion, we have attempted to make this report as self-contained as possible, in order to render it accessible and useful to the widest range of readers. Mathematical investigations of projectile motion have a rich and vital history, going back almost 500 years. Besides the obvious application to ballistics, there is a much more noteworthy connection (for our more pacific purposes) to developments and colourful personalities in mathematics and physics.

The aerodynamics characteristics of projectiles motion was therefore, a matter of national important and it received the attention of such distinguished scientists and engineers as Benjamin Robins, Galileo Galilei, Sir Isaac Newton, and Bash forth, Mayevski, Ingalls, Didion and many others. In etymology dictionaries the word ballistics; "art of throwing; science of projectiles", dates back as far as 1753 based on the Latin word ballista, a military machine for hurling stones as well as the Greek ballistes from ballein, "to throw; to throw as to hit". The art of ballistics is ancient, shaping projectiles such as arrows for a better, more predictable flight through air.

While projectiles were being launched all over the world the great minds of the time were trying to explain its mechanics and determine its motion. One particular person who studied in these fields was a man named Aristotle (384 BC–322 BC), a Greek philosopher and polymath. He argued that rest is the natural state of a body, so that some force that continues to act whilst the body remains in motion must accompany any movement. This force would be proportional to the velocity. This was the earliest recorded analysis of ballistic motion, known as ‟Impetus Theory‟ and was derived from Aristotelian dynamics. Various modifications to the Aristotelian Impulse Theory were put forward over many centuries in an attempt to bring together the mathematical analysis in line with observations. Finally in 1638

Galileo Galilei (15th February 1564–8 January 1642), an Italian physicist, mathematician, astronomer, and philosopher, who is also called „the father of Modern Science‟ published his paper named „Dialogues of the Two New Sciences‟‟, where it took on the question of projectile motion. Then, Sir Isaac Newton‟‟s book named Philosphiae Naturalis Principia Mathematica (Newton‟‟s Principia), published in 1687 was the theory of Gravitation along with his three famous laws of motion.Over the years many texts have discussed on projectile motions, include Moulton (1926), McShane et al (1953), Klimi (2008), Carlucci & Jacobson (2008), and R. McCoy(2012) and so on.

3. Physics of Projectile Motion

From general physics point of view with its definitions: It is study of the flight dynamics of projectiles through the interaction of the forces of propulsion, the aerodynamics of the projectile, atmospheric resistance, and gravity. It studies he patterns and relationships of the effects and characteristics of the physical environment over the free flight characteristics of the projectile.

The trajectory of a projectile is defined as a certain curve in space traced by the center of mass of a projectile in its flight under influence of drag and acceleration due to gravity through the air. A trajectory specifies the x, y and z coordinates of the center of gravity of the projectile at any time „t" and defined by: (a) x = x(t); (b) y = y(t); (c) z = z(t), where x(t), y(t) and z(t) are functions of the time t at t = 0. The trajectory is determined by: (a) the position of the origin; (b) the conditions of projections; (c) the characteristics of the projectile; (d) the characteristics of the air through which it passes. Mathematically the term trajectory refers to the ordered set of states (as a function of the time), which are assumed by a dynamical system over time. The Figure 3 (1a and 1b) depicts the 2D and 3D of the projectile motion.

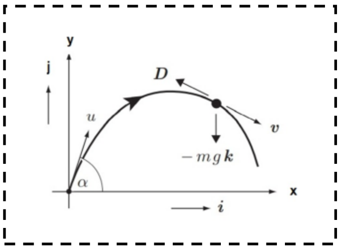

Fig. 3a: Motion in 2D

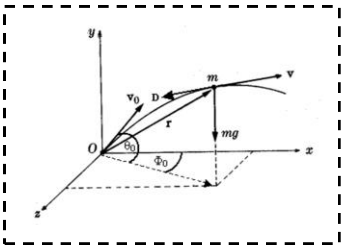

Fig. 3b: Motion in 3D

Figure 2: Geometry of Projectile Motion

3.1 Concept of Drag Force

The equation of motion for a body moving in medium in a vacuum, according to John Bernoulli and D'Alembert can be written as:

$$\frac{dv}{dt} = -kv^n - g \tag{1}$$

where k is a constant characteristics of the projectile. The Equation 1 is frequently called the fundamental equation of exterior ballistics. When n is an integer, and the differential equation can be solved for *v* as a function of t. The values of the exponent a for various velocities approximately as given below: (a) 0 <*v* < 250 m/s, n = 2; (b) 250 <*v* < 300 m/s, n = 3; (c) 300 <*v* < 375 m/s, n = 5; (d) 375 <v < 420 m/s, n = 3; (e) 420 <v < 550 m/s, n = 2; (f) 550 <v < 800 m/s, n = 1.7; (g) 800 <*v* <1200 m/s, n = 1.55. For constant atmospheric conditions, the sound speed is constant, and the Mach number and velocity are directly proportional. In this case, the drag coefficient (C_D) varies with Mach number and velocity in the same manner. The power-law fit shows excellent agreement over this Mach number range. For a specific projectile motion, the variation of C_D with Mach number M, in general, shows distinctly different characteristics in the three Mach number ranges: (a) subsonic, (b) transonic, and (c) supersonic. It can be demonstrated with a dimensional analysis that the aerodynamic drag coefficient, defined as $C_D = F/0.5\rho S v^2$ (quadratic case).

The magnitude of the drag force on an object is a function of the geometry of the

object, the density of the fluid, ρ, in which it travels, and proportional to the velocity of the projectile, *v*. Drag force is usually expressed as a function of these terms and a quantity known as a drag coefficient, C_D. In general the total drag force can be expressed according to object's speed as:

(a) For Linear Case (laminar flow): $F \propto v$ (2)

(b) For Quadratic Case (turbulent flow): $F \propto v^2$ (3)

Its definition differs depending on the body geometry. For most objects, the characteristic area is taken to be the frontal area. Discussing some of the basic properties of the resistive force, or drag, *f* (or F) of the air, or other medium, through which an object moves. We call it "air resistance" since air is the medium through which most projectiles move. In addition, the direction of the force due to motion through the air is opposite to the velocity v. As the linear drag force is much easier to solve mathematically, and we will consider with linear case because it is easier, and it allows us to introduce some useful mathematics. The function *f*(*v*) can be expanded by Taylor Series expansion about the point v=0, we have $f(v) = a + bv + cv^2 + \ldots$. The expression $f(v)$ can be seen as just an expansion of the drag force into its leading terms and the force $f(0) = 0$. For slow or high speeds, it is a good approximation for various applications. The Figure 4 shows a comparison of drag and non- drag profiles of a projectile motion.

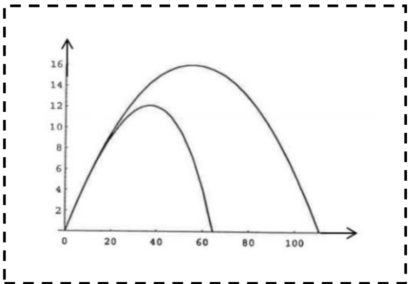

Figure 4: Drag and Non-drag

3.2 Assumptions for Projectile Motion

(a) The projectile is rigid and considered a material point of mass;

(b) Air resistance force direction coincides with the tangent on the trajectory, and the opposes to the direction of velocity;

(c) The acceleration due to gravity (g) is constant and its magnitude is 9.81 m/s^2; (d) Earth rotation is negligible and its surface is plane.

(e) All other types of forces & moments, like centrifugal, coriolis and its cross effects are negligible;

(f) Projectile moves in the static and thermodynamic stable atmosphere (no wind and with a standard changes of parameters);

(g)No chaotic behaviours at initial position;

3.3 Equations of Projectile Motion

The motion of a projectile is generally thought of as being governed by a system of ordinary differential equations relating its position, velocity, and acceleration to the system of forces acting on it. For a simple case, this paper discusses a projectile motion with two-dimensional (2-D) case. The projectile considers as a particle with mass m influenced by gravity g and air resistance. Therefore the trajectory modelling of the projectile motion can be described as:

$$a_x = \frac{d^2x}{dt^2} = -(1/m).F_D.\cos\theta \; ; \; a_x = \frac{d^2y}{dt^2} = -(1/m)\cdot(F_D \cdot \sin\theta + m\cdot g) \qquad (4)$$

where $v = \sqrt{v_x^2 + v_y^2}$, $\theta = \tan^{-1}(v_y / v_x)$ and $F_D = 0.5\times\rho\times v^2\times S\times C_D$

The above equations can be solved subject to the following state or initial conditions :

$$u = dx/dt = v_0\cos\theta_0 \; ; \; v = dy/dt = v_0\sin\theta_0 \; ;$$

$$v = v_0 \; ; x = y = 0 \; ; \text{ at } t = 0 \qquad (5)$$

where the drag force F_D acting on the projectile depends on air density ρ, velocity *v*, cross- section area *S*= $\pi\times r^2$, radius of the projectile *r*, and drag coefficient C_D. If the drag force is neglected, the calculation of the projectile trajectory becomes trivial. If however the drag force is taken into account, the analytical solution is not soluble due to the drag dependency on the square of the velocity and presence of angle θ as argument of trigonometric functions.

3.4 Computation of Motion Equations

An ordinary differential equation of motion involves one or more functions of one independent variable. A typical application is to systems of particles: the positions and velocities of the particles are functions of one variable, the time. In order to compute the motion of a projectile a numerical integration method is needed. During each time step in the computation all exterior forces are calculated and then transformed into accelerations, which used in order to determine the current velocities and the new position of the projectile. In this paper the basic approach of numerical integration is Euler's forward method

$$y_{n+1} = y_n + hf\,(t_n, y_n), \qquad (6)$$

where y_{n+1} is the next value, y_n is the current value, h is the time step, and $f(t_n;y_n)$ is the current derivative. Based on the current value, time step and derivatives, Euler's forward method anticipates the next value by adding the product of the current derivative and the time step to the current value. For short computations with a small time step, Euler's forward method can be sufficient.

3.5 MATLAB Source Codes for Computation **(**ProjectileMotion.m**)**

```
%ProjectileMotion.m
%Calculation of position, velocity, and acceleration for a projectile in
%motion with air resistance versus time.
%The equations of motion are used for small time interval clear;
%NPTS=200;                      %example maximum number of points
%TMAX=20.0;                     %example maximum time
%
TTL=input(' Enter the Title Name TTL:->','s');     %string input
%
NPTS= input(' Enter the Number Calculation Steps desired NPTS:->' );
%
TMAX= input(' Enter the Run Time TMAX:->  ');
%
NT=NPTS/10;                              %to print only every NT steps
%
%g=9.81;m=1.0;c=0.05;y0=0;v0=110;    % parameters
%
g=input('Enter  value of gravity g(m/s^2):->   '); m=input('Enter  the value of
mass m(kg):->  '); c=input('Enter the value of drag coefficient  c:->  ');
y0=input('Enter  the  initial  height  y0(m):->     ');               %Initial  Conditions
v0=input('Enter  the  initial  velocity  v0(m/s):->    ');           %Initial  Conditions
FLAG=input('  Enter  0 (v drag) or 1(v^2 drag) FLAG:->  ');
t0=0.0;                                           %start at time t=0
dt=TMAX/NPTS;                                     %time step size
if  FLAG ==0
F= -m*g-c*v0;                                   % initial  force  - case 1
vt=abs(m*g/c);                                  % terminal velocity elseif
FLAG==1
F=-m*g-c*v0*abs(v0);                            % initial force – case 2
vt=sqrt(m*g/c);                                 % terminal velocity
end;
%dt, FLAG, and vt used
fprintf(' FLAG=%1i, Time step dt=TMAX/NPTS=%5.2f, vt=%5.2f\n',FLAG,dt,vt);
a0=F/m; %initial acceleration
```

```
fprintf('      t           y            v            a\n');  %output  column  labels v(1)=v0;
y(1)=y0; a(1)=a0; t(1)=t0;
fprintf('%7.4f   %7.4f   %7.4f  %7.4f\n', t(1),y(1),v(1),a(1)); %print initial values
for i=1:NPTS
v(i+1)=v(i)+a(i)*dt;                                        %new velocity
y(i+1)=y(i)+v(i+1)*dt;                                      %new position
t(i+1)=t(i)+dt;                                             %new time
if   FLAG==0
     F=-m*g-c*v(i+1);                                       %new force – case 1
elseif  FLAG==1
F=-m*g-c*v(i+1)*abs(v(i+1));                            %new force – case 2
end a(i+1)=F/m;
%print only every NT steps if(mod(i,NT)==0)
   fprintf('%7.4f   %7.4f   %7.4f  %7.4f\n', t(i+1),y(i+1),v(i+1),a(i+1));
   end;
end;
plot(t, y,'k-',t, v, 'b:', t, a, 'r-.');
ylabel('y(m), v(m/s), a(m/s^2)', 'FontSize',10);
xlabel('time(s)','FontSize',10); grid on title(TTL,'FontSize',10);
h=legend('position','velocity','acceleration',0); set(h,'FontSize',10)
%
```

3.6 Inputs for Computation

In this paper, the three sets of input parameters as per MATLAB source code are selected for the computation or simulation purpose to compute the equations of the projectile motion. For example the format of one set as: (a) Enter the Title Name TTL:->Projectile Motion with Air Resistance; (b) Enter the Number Calculation Steps desired NPTS:->200; (c) Enter the Run Time TMAX:-> 20; (c) (d) Enter value of gravity g(m/s^2):-> 9.81; (e) Enter the value of mass m(kg):-> 1.0; (f) Enter the value of drag coefficient c:-> 0.05; (g) Enter the initial height y0(m):-> 0.0; (h) Enter the initial velocity v0(m/s):-> 110; (i) Enter 0 (v drag) or 1(v^2 drag) FLAG:-> 0. The users can select other set of input parameters as per code format.

3.7 Outcomes of Projectile Motion

The Figure 5 shows the graphical outputs are depicted as the following figures.

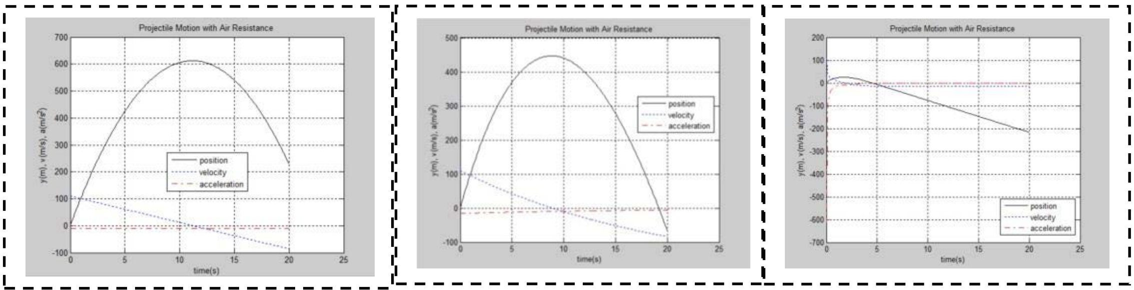

Fig. 5a: Zero Drag Force **Fig. 5b:** Linear Drag Force **Fig. 5c:** Quadratic Drag Force

Figure 5: Trajectory Profiles of a Projectile Motion

4. Conclusion

The study of the motion of bodies has been one of the longest and most intensive studies of mankind. The projectile motion via external ballistics is that branch of applied physics which deals with the motion of projectiles and the conditions governing that motion. Its motion in air resistance has a wide variety of applications in the industries, civilians and particularly military. This paper presents an approach for analysis of a projectile motion. The motion equations have been computed for a set of parameters. In the limited time frame, the three dimensional concept has not been not considered of the projectile all aspects by incorporating all parameters.

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