Orissa Journal of Physics
ISSN 0974-8202

Vol. 26, No.2
August 2019
pp. 129-136

# Role of radius on magnetic field of a solenoid of finite length revisited

S N MAITRA

*Retired Head of Mathematics Department*
*National Defense Academy, Khadakwasla, Pune-411023*

*Received: 12.6.2019* ; *Revised : 5.7.2019* ; *Accepted : 25.7. 2019*

**Abstract :** The role of radius on the magnetic field of a solenoid at an axial point is vividly studied. First, it is established that with given length of a solenoid its magnetic field at given axial point outside the solenoid has a maximum with respect to an optimum value of its radius. Variation of the magnetic field at any axial point inside/outside the solenoid with its radius with given surface area/ volume of the solenoid is also studied. Magnetic field of a solenoid of considerably large length at any axial point is formulated in two different cases and the curve depicting the distribution of the axial magnetic field and radius is found to be a parabola.

## 1. Introduction

V.P. Srivastava[1] studied variation of magnetic field of a solenoid of finite length with its radius graphically and with numerical examples.VN Shukla[2] dealt with magnetic effects of electric current.SY Gambhir[3] solved some problems related to electromagnetic induction. In this article is studied the same aspect but analytically to yield some results with new insights, e.g., herein is also derived a formula for magnetic field due to a solenoid of considerably greater length compared to its radius. This formula involves both length of the solenoid and its radius and has knowledgeably not yet found its way into the literature.

Further, with given (1) volume of the solenoid, (2) surface area of the solenoid and (3)length and diameter of the solenoidal wire, we have studied the effect of radius on the axial magnetic field.It has been also established that the magnetic field of a solenoid at an axial point outside the solenoid attains a maximum for an optimum value of its radius, with given other parameters.

## 2. Formulation of axial magnetic field due to a lengthy solenoid

The magnetic field at the centre of an infinitely long solenoid with current i, magnetic permeability μ and number of turns n as found in textbooks on Electricity and Magnetism is

$$B_0 = \mu_0 ni \tag{1}$$

which is independent of the radius of the solenoid. The magnetic field[1] at a point, z distance, from the centre along its axis inside or outside the solenoid of length 2l (=L) and radius a is

$$B=\frac{B_0}{2}\left[\frac{l+z}{\sqrt{a^2+(l+z)^2}}+\frac{l-z}{\sqrt{a^2+(l-z)^2}}\right] \tag{2}$$

which more specifically suggests that magnetic field at any axial point inside the solenoid is

$$B=\frac{B_0}{2}\left[\frac{l+z}{\sqrt{a^2+(l+z)^2}}+\frac{l-z}{\sqrt{a^2+(l-z)^2}}\right] \quad \text{for } z\leq l \tag{3}$$

whereas the magnetic field at such a point outside the solenoid is

$$B=\frac{B_0}{2}\left[\frac{z+l}{\sqrt{a^2+(l+z)^2}}-\frac{z-l}{\sqrt{a^2+(l-z)^2}}\right] \quad \text{for } z\geq l \tag{4}$$

Equation (2) reveals that as the radius increases the magnetic field at a particular point inside the solenoid decreases with given length 2l and parameter $B_0$. But from equation (3) we cannot ascertain this kind of variation of the axial magnetic field outside the solenoid due to variation of the radius.

## 3. Variation of magnetic field with radius subject to some given parameters

With given volume of the solenoid ,ie, $V=2\pi a^2 l$, equation (2) and (3) can respectively be written as

$$B=\frac{B_0}{2}\left[\frac{1}{\sqrt{1+\frac{1}{\frac{1}{a^2}\left(\frac{V}{2\pi a^2}+z\right)}}}+\frac{1}{\sqrt{1+\frac{1}{\frac{1}{a^2}\left(\frac{V}{2\pi a^2}-z\right)}}}\right] \tag{5}$$

Which confirms that with given volume of the solenoid, such magnetic field inside the solenoid decreases as the radius increases in that both the terms on the right-hand side of (5) then decrease in either situation.

With the same given parameter as above, equation (2) or (4) can be rewritten:

$$B=\frac{B_0}{2}\left[\frac{1}{\sqrt{1+\frac{1}{\frac{1}{a^2}\left(\frac{V}{2\pi a^2}+z\right)}}}-\frac{1}{\sqrt{1+\frac{1}{\frac{1}{a^2}\left(z-\frac{V}{2\pi a^2}\right)}}}\right] \qquad (6)$$

Regarding the right-hand side of (6) as difference between the two positive terms,it is observed that as the radius increases, the first term decreases and the second term without the negative sign either decreases or increases depending on the aforesaid parameters.

With given surface area of the solenoid ,ie, $S=4\pi al$, equation (3) and (4) can be rewritten as

$$B=\frac{B_0}{2}\left[\frac{1}{\sqrt{1+\frac{1}{\frac{1}{a^2}\left(\frac{S}{4\pi a}+z\right)}}}+\frac{1}{\sqrt{1+\frac{1}{\frac{1}{a^2}\left(\frac{S}{4\pi a}-z\right)}}}\right] \qquad (7)$$

$$B=\frac{B_0}{2}\left[\frac{1}{\sqrt{1+\frac{1}{\frac{1}{a^2}\left(z+\frac{S}{4\pi a}\right)}}}-\frac{1}{\sqrt{1+\frac{1}{\frac{1}{a^2}\left(z-\frac{S}{4\pi a}\right)}}}\right] \qquad (8)$$

leading to the same kind of variation of the magnetic field with the radius inside and outside the solenoid as in case of its given volume.

## 4. Formulation of magnetic field due to considerably large solenoid

If the solenoid is so large that

$l-z\geq a$ in equation (2), by Binomial theorem neglecting squares and other higher powers of $\frac{a}{l-z}$ and $\frac{a}{l+z}$, this equation turns out to be

$$B=\frac{B_0}{2}\left[1-\frac{a^2}{2(l+z)^2}+1-\frac{a^2}{2(l-z)^2}\right]$$

Or, $$\frac{B}{B_0}=1-\frac{a^2(l^2+z^2)}{2((l^2-z^2)^2} \qquad z\leq l \qquad (9)$$

which reveals that the curve representing the distribution of the magnetic field at an inside- axial point with radius of the solenoid is parabola and can be put in non dimensional form

$$y=1-x^2 \qquad (10)$$

where $x=\frac{a\sqrt{(l^2+z^2)/2}}{l^2-z^2}$ and $y=\frac{B}{B_0}$ (11)

which give the non dimensional radius and non dimensional magnetic field at given point.

The foregoing problem can be tackled in more approximate and rigorous method for a solenoid in which a<<l+z together with $a^2+(l-z)^2$>>4lz.

Then neglecting the relevant squares and other higher powers in the Binomial expansion in equation(2) we get

$$B=\frac{B_0}{2}[\frac{l+z}{\sqrt{a^2+(l+z)^2}}+\frac{l-z}{\sqrt{a^2+(l-z)^2}}(1-\frac{4lz}{a^2+(l-z)^2})^{-\frac{1}{2}}]$$

$$=\frac{2lz}{(l+z)[1+\frac{a^2}{(l+z)^2}]^{\frac{1}{2}}}+\frac{2(l-z)lz}{(l+z)^3[1+\frac{a^2}{(l+z)^2}]^{\frac{3}{2}}}$$

$$=\frac{2lz}{(l+z)}\left[1-\frac{a^2}{(l+z)^2}\right]+\frac{2(l-z)lz}{(l+z)^3}\left[1-\frac{3a^2}{2(l+z)^2}\right]$$

$$\frac{B}{B_0}=\frac{2l^2(l+3z)}{(l+z)^3}-\frac{l(5zl-2z^2+l^2)a^2}{2(l+z)^5} \tag{12}$$

which in non dimensional form is also a parabola of equation

$$Y=A-X^2 \tag{13}$$

where the dimensionless radius X, dimensionless magnetic field Y and constant A are respectively given by

$$X=\sqrt{\frac{2l(5zl-2z^2+l)}{2(l+z)^5}}\,a\ ,\quad Y=\frac{B}{B_0}\ \text{and}\ A=\frac{2l^2(l+3z)}{(l+z)^3} \tag{14}$$

That is why curve[1] appears to be almost a parabola. Putting z=l and z=0 respectively in equation (12) we get the equations of the magnetic field at the end points of the solenoid:

$$Y=1-\frac{a^2}{8l^2} \tag{15}$$

$$Y=1-\frac{a^2}{2l^2} \tag{16}$$

The magnetic field at the centre of the solenoid can also found out by a similar treatment , ie ,by putting z=l and z=0 in equation (2),we obtain magnetic fields respectively at either end point and centre of the solenoid. Including the higher powers of $a^2$ involved in equation after Binomial expansion we get with more accuracy the magnetic field as a polynomial function of the radius of the solenoid.

## 5. Formulation of the magnetic field outside the solenoid

If the dimensions of the solenoid are such that z-l>>a, then neglecting the squares and other higher powers of $\frac{a^2}{(z\pm l)^2}$ in the Binomial expansion in (4),

$$\frac{B}{B_0}=\frac{1}{2}[1-\frac{a^2}{2(z+l)^2}-1+\frac{a^2}{2(z-l)^2}$$

Or, $$\frac{B}{B_0}=\frac{a^2lz}{2((l^2-z^2)^2} \quad (17)$$

Hence putting $\frac{B}{B_0}=Y$ and $\frac{a\sqrt{lz}}{l^2-z^2}$ = X in (17) we get the equation of the magnetic field at an outside- axial point:

$$Y=X^2 \quad (18)$$

which is a standard parabola with its vertex at the origin .

## 6. Variation of the axial magnetic field with radius of the solenoid for given dimension of the solenoid wire

Let us suppose Length of the solenoid wire=$L^1$, its diameter=d, number of turns of the wire wound on the solenoid = n. Then the following relationship will hold:

$$2\pi a\frac{2ln}{d}=L^1$$

$$n=L^1d/(4\pi al) \quad (19)$$

and then (1) gives

$$B_0=\mu_0\left(\frac{L^1d}{4\pi al}\right)i \quad (20)$$

using which in (2) one gets

$$B=\mu_0\left(\frac{L^1id}{8\pi al}\right)[\frac{l+z}{\sqrt{a^2+(l+z)^2}}+\frac{l-z}{\sqrt{a^2+(l-z)^2}}] \quad (21)$$

which reveals that the axial magnetic field inside the solenoid increases with the increase in diameter of the solenoid wire and with the decrease in the radius of the solenoid.

## 7. Maximization of the magnetic field at an outside- axial point with respect to radius of the solenoid

For convenience putting $a^2=b, z+l=x_1$, $z-l=x_2$, (22)

In relationship (4), one gets

$$B=\frac{B_0}{2}\left(\frac{x_1}{\sqrt{b+x_1^2}}-\frac{x_2}{\sqrt{b+x_2^2}}\right) \quad (23)$$

For maximum or minimum of magnetic field B with respect to the radius ,ie b we have from (23)

$$\frac{dB}{db}=0=\frac{-B}{4}\left[\frac{x_1}{(b+x_1^2)^{\frac{3}{2}}}-\frac{x_2}{(b+x_2^2)^{\frac{3}{2}}}\right] \quad (24)$$

$$x_1(b+x_2^2)^{\frac{3}{2}}=x_2(b+x_1^2)^{\frac{3}{2}}$$

$$x_1^{\frac{2}{3}}(b+x_2^2)=x_2^{\frac{2}{3}}(b+x_1^2)$$

$$b=\frac{x_1^2x_2^{2/3}-x_2^2x_1^{2/3}}{x_1^{2/3}-x_2^{2/3}}$$

$$b=(x_1x_2)^{\frac{2}{3}}(x_1^{\frac{2}{3}}+x_2^{\frac{2}{3}}) \quad (25)$$

Further $\frac{d^2B}{db^2}=\frac{3B_0}{8}\left[\frac{x_1}{(b+x_1^2)^{\frac{5}{2}}}-\frac{x_2}{(b+x_2^2)^{\frac{5}{2}}}\right]=\frac{-3B_0lzk}{(b+x_1^2)(b+x_2^2)}<0$

where from (24)

$$k=\frac{x_1}{(b+x_1^2)^{\frac{3}{2}}}-\frac{x_2}{(b+x_2^2)^{\frac{3}{2}}}>0 \quad (27)$$

Equations (25) to (27) indicate that this outside axial magnetic field that occurs F

$$a=\sqrt{(z^2-l^2)^{\frac{2}{3}}\{(z-l)^{\frac{2}{3}}+(z+l)^{\frac{2}{3}}\}} \quad (28)$$

can be given by

$$B_{max}=\frac{B_0}{4\sqrt{zl}}\{(z-l)^{\frac{2}{3}}-(z+l)^{\frac{2}{3}}\}^{3/2} \quad (29)$$

## 8. Verification of existence of maximum magnetic field at an outside axial point for an optimal radius of the solenoid

In the light of the forgoing theory let us consider solenoids of different radii but of the same length 2l and outside axial point at distance z along with given $B_0$. As such let us take numerically $x_1=l+z=6$ cms and $x_2=l-z=2$ cms and $Y=$ dimensionless magnetic field $=\frac{B}{B_0}$ so that equation (4) gives

$$Y=\frac{3}{\sqrt{a^2+36}}-\frac{1}{\sqrt{a^2+4}} \quad (30)$$

Now we prepare the following table giving the distribution of the magnetic field Y at outside axial point with monotonic increasing radius a of the solenoid.

**Table 1**

| Radius a | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $a^2$ | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 | 121 |
| Magnetic field outside Y | .046 | .120 | .169 | .193 | .198 | .195 | .189 | .179 | .173 | .171 | .164 |

Table1 shows that with given parameters an outside-axial point magnetic field attains a maximum when the radius of the solenoid is 5cm. Interestingly this maximum magnetic field is analytically obtained from formula (29)

$$Y_{max} = \frac{1}{4\sqrt{zl}}\{(\mathrm{z}+\mathrm{l})^{\frac{2}{3}} - (\mathrm{z}-\mathrm{l})^{\frac{2}{3}}\}^{3/2}$$

$$\frac{1}{4\sqrt{4\times2}}\{(6)^{\frac{2}{3}} - (2)^{\frac{2}{3}}\}^{3/2} \quad (31)$$

=.199

And from (28) the optimum radius is

$$a_{0pt}=(6\times2)^{1/3}\{(6)^{\frac{2}{3}} + (2)^{\frac{2}{3}}\}^{1/2} = 5.03 \quad (32)$$

Hence with negligible errors, (31),(32)and table1 confirm the existence of the above maximum magnetic field.

## 9. Discussion and conclusion

In essence we have embodied two different functions of the radius to determine the magnetic field of the solenoid at its inside-axial point and outside-axial point respectively. Since the magnetic field is a continuous function of the radius, the magnetic field at either axial-end point (z=l) of the solenoid is the same by use of both the formulae, ie, by taking the limiting case $z \to l$ where z is the distance of the end point from the centre of the solenoid.

For a considerably large solenoid whose length is considerably larger than the radius, inside- axial-point magnetic field versus radius of the solenoid is parabola, illustrated in Figure 1, whereas the functional relationship of the outside-axial point- magnetic field with radius of the solenoid is a standard parabola, vide Figure 2.But with no restriction on the length vis-à-vis its radius, at every outside point there exists a magnetic field which is maximum for an optimum radius of the solenoid with given other parameters. In this context table1 reveals that this magnetic field increases with the increase in radius starting from the unity and thereafter attains the maximum value for a certain value of the radius, with further increase in the radius, the magnetic field decreases.

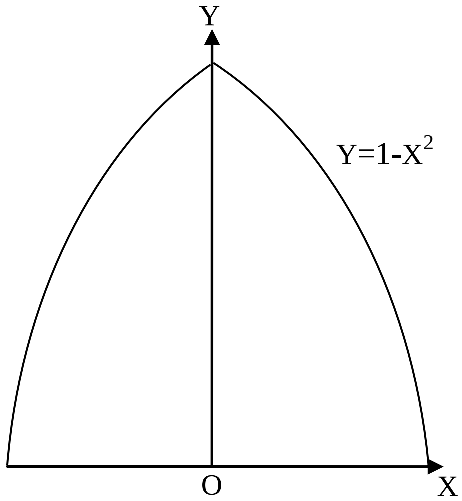


**Fig.1:** Graph of inside-axial-point magnetic field versus radius in case of solenoid of length larger than its radius is an inverted parabola.

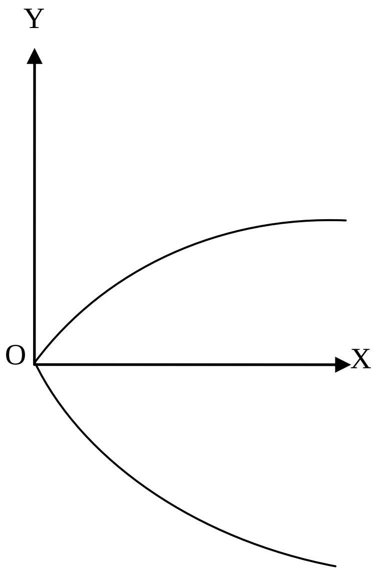


**Fig.2:** Graph of outside-axial-point magnetic field versus radius of the solenoid is a standard parabola.

Contrary to the forgoing formulations and analyses there is a point[1] which is amended as that variation of the magnetic field inside the solenoid is very small for certain range of the radius depending on the length of the solenoid and distance z.

## References


[1] VP Srivastava, *Bulletin of Indian Association of Physics Teachers*, June,1997.

[2] VN Shukla and et al, *A Text Book of Fundamental Physics*, (Nirali Prakashan, Pune, 2007)

[3] SY Gambhir, *Narendra Problems in Physics*, Narendra Prakashan, Pune, 1995)