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Parametric Resonance as Explanation of Mode-locking in Acoustics

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Abstract. Parametric resonance is a selective topic in mechanics with very limited instances of application. Of course, there are case examples of electric circuit and electromagnetic phenomena, mentioned in literature. There are recent reports of identification of occurrence of parametric resonance in acoustics, in the well-known Melde's Experiment and in passage of ultra sonic vibrations in fluid filled cavity. In our attempt to study the acoustics of conch shell, we observed the locking in of sound at odd harmonics of the fundamental and repeated our experiments with cylindrical tubes to confirm the finding. Although mode-locking has been mentioned in classic texts like Philip Morse and N. Fletcher, explanation of the phenomenon looks lacking in literature. Therefore, in this paper we put-forth the theory of parametric resonance as an explanation of this well-known acoustic phenomenon. Vis-à-vis, we present a brief review of the literature to substantiate our stand. Further it is worth pointing out that existence of threshold driving amplitude, as observed in the experiment, ensues as a natural outcome of this formalism.

Keyword. Parametric Resonance; Mathieu's equation; range of instability; mode locking; threshold driving amplitude.

PACS: 43.20.-f, 43.20.Bi, 43.20.Ks, 43.75.−z, 43.75.Np

1. Introduction

Parametric resonance is dealt in somewhat advanced course in mechanics. It is said to occur[1,2] when one or both of basic parameters of an oscillator, namely mass and spring constant or their analogues are appreciably affected by the external cause, eventually becoming time dependant. Therefore, the frequency of the system effectively turns out to be time-dependent. This is precisely presented by the oscillator equation.

$$\ddot{x} + \omega_0^2(t)x = 0 \tag{1}$$

One may add a damping component to equation (1), if frictional dissipation is present. Instances of case of resonance arising out of the solution of equation (1) are rather limited and occur in different branches of the physics. A couple of frequently cited examples in texts on mechanics [1,2] are (i) a simple pendulum whose support of suspension yields by executing vertical oscillation, there by affecting the acceleration due to gravity and consequently its frequency and (ii) the pumping in of a swing which is started oscillating by a by-stander with a small amplitude and then the rider amplifies the amplitude by alternately sitting and rising.

It also occurs in axially exited beams, slashing liquids and in parallel tuned circuits with time-varying capacitance [3]. Further, literature points out that parametric excitation has been observed in mechanical systems coupled to magnetic or electric components[4] and cites several references on these themes, in particular, parametric magneto-elastic resonance of a perfect elastic conductor in magnetic field[5]. In a similar system, the interaction between a time-dependant magnetic field and elastic beam plate can lead to unstable vibration under certain magnetic field fluctuation frequencies[6]. Parametric resonance occurs also in acoustics in nonlinear string vibration as dealt by Rowland [7] and in passage of ultrasonic vibration in fluid filled cavity[8,9]. A more relevant report in our context is a work by Shyh Wang et.al[10]. dealing with parametric generation of acoustics wave and mode-locking in spin-waves through magneto elastic coupling. Even the nature of vibration of string in the most familiar Melde's experiment analyzed by Lord Rayleigh and C.V.Raman [11,12] has been identified rather recently [7] as a case of parametric resonance.

In our study of sound spectrum produced by lip-driven conch shell, we observed that the spectrum invariably contains peaks at $3f_0$ or $5f_0$. f_0 being the fundamental frequency, along with over-tones at frequencies which are all integral multiples of the peak frequency[13]. In order to confirm our observation we repeated our experiment with lip-driven pipe and horn where the spectra were again similar to the case of conch-shell. Such skipping of resonance to a higher over-tone is called mode-locking in literature. It is a well-known phenomenon in acoustics[14,15]. However, a satisfactory explanation of the effect is not found in literature. In the following section we present a brief review of such explanations. Hence in this paper we attempt to explain the phenomenon as a case of parametric resonance. Our formulation also provides the explanation for occurrence of threshold driving amplitudes for locking in at different frequencies.

In section-2 we present a brief review of different attempts for explanation of mode-locking from literature. Section-3 carries our parametric formalism for mode-locking. In section-4 we obtain the regions of instabilities for parametric resonance at different frequencies. Section-5 deals with the origin of threshold driving amplitude required for causing mode-locking at definite frequency. Section-6 carries our conclusion.

2. Mode-locking: Its explanations

Giving his observation on mode-locking Phillip Morse in 1940's writes [14] "When the pipe is blown more strongly, the jet frequency is first held near the fundamental free frequency of the tube by the strength of coupling, but when the edge tone by itself would exhibit a frequency close to the third harmonic of the pipe, the note suddenly changes to this over-tone and "locks in" at the new frequency. The pipe is then said to be "overblown". As indicated XXX only the odd harmonics are present to any extent in the sound from a closed pipe; the dependence of the amplitude of the higher harmonics on the dimension of the driving jet of air and on the location and shape of the "lip" regulating the edge-tone is too complex XXX". However, somewhat detailed analysis of this 'complex' effect is found in Fletcher's series of works[15,20].

Its origin may be trace is traced in the interaction of the blown jet with the pipe mode at the pipe-lip and is attributed to jet - driven nonlinearly. The velocity profile of the jet is concluded to tend towards a smooth bell-shaped form that can be described by[15,16]

$$V(z) = V_0 \operatorname{sech}^2\left(\frac{z}{b}\right) \qquad (2)$$

,

where V_0 is the jet center- plane velocity and *z*, the co-ordinate transverse to the jet. Here *b* is a scale factor determining the width of the profile, which varies along the jet. If *h* be the jet tip displacement, and h_0, the lip offset from the undisturbed center plane of the jet, the expression for flow of jet, U_j, as a function of V, *h* and *b* is given by

$$U_j = w\int_{-\infty}^{h_0} V_0 \operatorname{sech}^2\left(\frac{z-h}{b}\right)dz = wbv_0\left\{\tanh\left(\frac{h_0-h}{b}\right)+1\right\}, \quad (3)$$

where *w* is the jet width in its transverse plane. The shape of jet flow is graphically demonstrated in the text to match with the given description. Excitation modes dependent upon the lip-offset is also depicted as relative level of harmonics in an organic pipe. Such conditions may lead to quenching of certain harmonics and appearance of some other. This qualitatively accounts for the mode locking.

Further, a more rigorous mathematical treatment of the phenomenon is found in Fletcher's 1978 work[19]. It is stated at the outset that in-harmonically related sounds can be produced on most wind instruments, whereas their normal tones may sound with accurate harmonics, locked in both phase and frequency to the fundamental. And the paper aims at tracing out the features which cause mode 'locking'. In order to account for that, the author puts forth a general treatment, which is defined to be valid for both string and wind instruments.

The basic equation for the i^{th} mode is given by

$$\ddot{x}_i + k_i\dot{x}_i + \omega_i^{\,2}x_i = \lambda_i F\left(\dot{x}_j\right), \quad (4)$$

where x_i is the generalized co-ordinate associated with the i^{th} mode and $\dot{x}_i$,the corresponding generalized velocity, λ_i is the coupling strength. $F\left(\dot{x}_j\right)$ is the generalized velocity dependent driving force. In putting forth the mechanism, the author argues that there is a phase delay δ_i consequent due to the blow- jet & the pipe mode interaction. This phase delay is built into $\dot{x}_i$ and $F\left(\dot{x}_j\right)$, identified as the acoustic flow velocity and driving pressure respectively.

In order to solve equation (4), Fletcher implements Bagoliubov-Mitropolsky's technique to finally arrive at the mode locking condition,

$$p\dot{\phi}_i - q\dot{\phi}_j = q\omega_j - p\omega_i \, , \tag{5}$$

where ϕ and ω are the phase and frequency of the respective modes. Although the exercise is exhaustive, it does not clearly expose the conclusion of mode locking to satisfactorily explain the observation.

Beside, at the concluding part of the paper, in way of discussion, the author specifies a number of conditions as requirement for mode locking. They are,

1. The integers p and q specifying the relations between the modes must be small, i.e. $p + q < 4$
2. The sounding frequencies and the mode frequencies must be harmonically related.
3. The Coupling between the two modes and the driving force must be large.
4. The driving force must be highly non-linear.
5. The mode amplitude must be large.

Out of these requirements, although condition (2) to (5) might be qualitatively borne out in blowing of pipes and conch shell, one of the most significant numerical condition (1) is clearly not valid for locking of 5th mode, where for the fundamental mode $p = 1$ and for the locked in mode, $q = 5$, so that $p + q = 6$ and greater than 4.

One more important fact not explained by any of the existing theory is that, the 'locking-in' occurs, in blowing pipes and conch shell, as we have observed, at an odd multiple of apparatus fundamental of which, samples of spectrum and data we present in the following (figure 1 a-c, and table-1).

Certain notes on the data in the table and their process of collection are in order here. In the experiment, sound from lip driven pipe was received with microphone and analyzed with sound technology FFT software (Spectra Plus). Only three slides of spectrum view of sound from lip-driven pipe of 110 cm. length and diameter 1.9cm are presented here as sample for inspection. Therefore, the data table-1 carries observation of peak frequencies of sound from lip-driven pipes of 11 different lengths and blown at varying pressures. All the frequency data are collected from computer display, as exhibited in the sample spectral views.

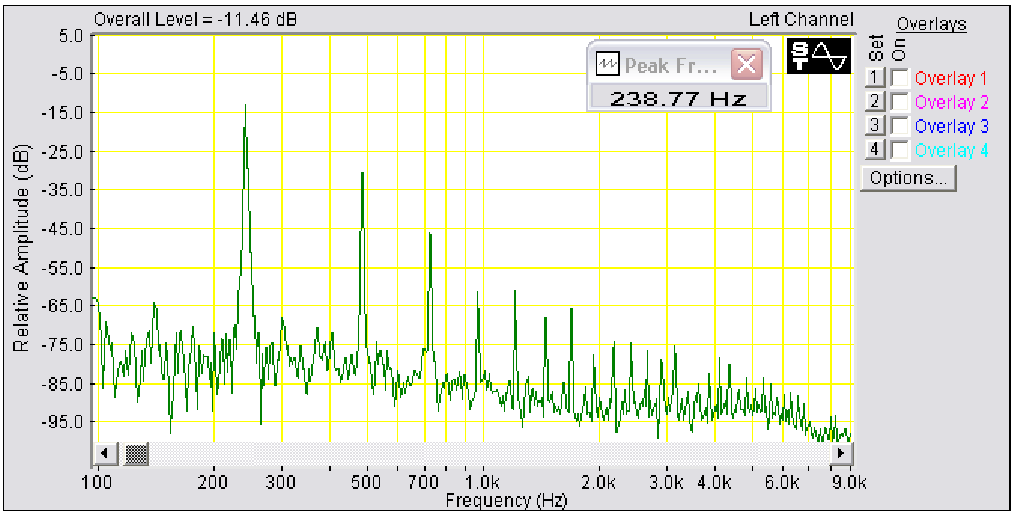

Fig. 1(a). Spectrum view of lip driven pipe of length of 110cm. & diameter 1.9cm.

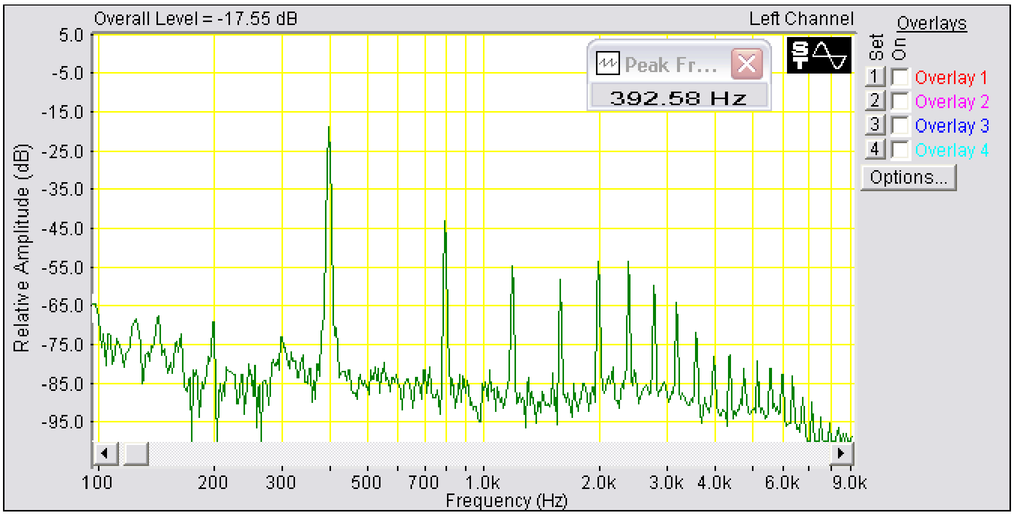

Fig. 1(b). Spectrum view at higher pressure with shifted frequency at the same tube length of 110 cm.

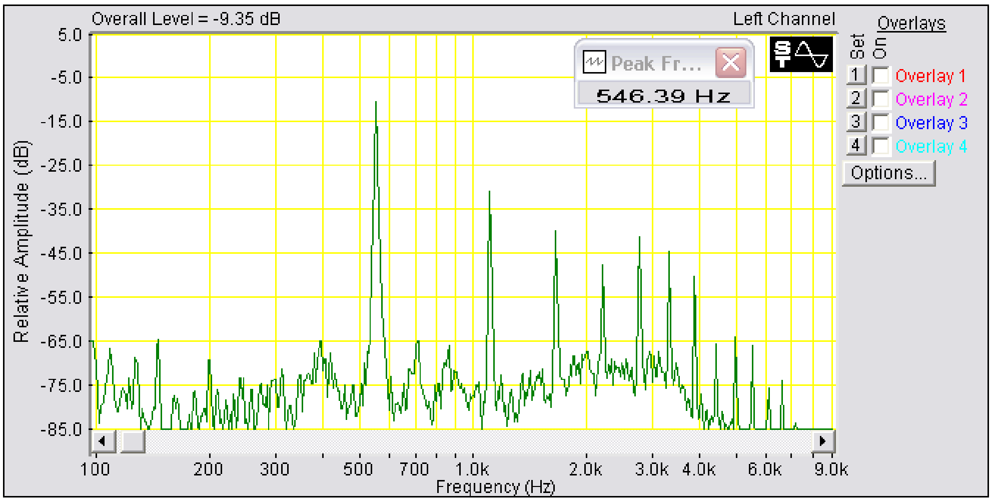

Fig. 1(c). Spectrum view at more higher pressure with shifted frequency at the same tube length of 110 cm.

Table 1 . Measured peak frequency for different length and diameter of pipes.

Length of Pipe(l) in cm.	*C/4l in Hz*	Measured peak frequency in Hz.		
		Diameter of pipe 1.3cm.	Diameter of pipe 1.9cm.	*Diameter of pipe 2.5cm.*
150	57.5	291.5,398.4	291.5,395.51	172,290
140	61.6	312.01,437.3	312,430	183.1,309
130	66.3	335,471.60	333.98,455.57	200, 332
120	71.8	363,503.90	216,354,496	216, 360
110	78.4	394,553	238.77,392.58,546.39	244,396
100	86.2	254,435	262,421	262.2,432
90	95.8	288,483	287.11,468.75	96.2,290
80	107.8	319.3,541	328,540	109,322
70	123.2	375,608	366.21,580.08	120.12,360.35
60	143.7	436.5,719	424,709	140,420
50	172.5	517.09,861	177.25,511.23	175,512

* C= 345m./sec is the velocity of sound and *l* is the length of tube.

One should not be tempted to jump into a simple-minded conclusion at this stage that the odd multiple of the 'lock-in' frequency occurs due to the close-open nature of the tube. Details of the spectra on scrutiny show that in lip-driven wind instrument like the pipes, horns and conch shell, the spectra exhibit overtones with frequencies, which are all integral multiples of the lock-in frequencies. This compels us to suggest that the lip-driven instruments simultaneously behave as closed-open and open-open system. This technically amounts to rapid change of the impedance sinusoidally at the tube-lip[2].

Therefore, in order to explain our observations of 'locking-in' in pipes, conch shell and elsewhere, we present in the next section, an alternative and relatively simpler theory of the phenomenon in light of parametric resonance.

3. Parametric Formalism for Mode Locking

Parametric resonance has been mentioned in the introduction to follow equation (1), involving a time dependent frequency ω(t), through time dependence of both or either of the system parameters of inertia, *m* and the restoring constant, *k*. Here, in case of a resonating column of air in pipe or horn, the independent parameters are bulk modulus K and density ρ of the air mass. Excitation of a tube or horn by blowing affects the density of air mass in the cavity as an instantaneous effect. If the blowing pressure is time dependent, it causes the air density time dependent too. It is a well known result of the theory of resonant cavity that the frequency $f = \frac{1}{2\pi}\sqrt{\frac{K}{\rho}}$. If ρ is ρ (t), the frequency ω is also time dependent i.e. ω = ω(t). Hence parametric resonance may occur in a resonant cavity.

The form of ω(t) is usually decided by the conditions of the problem. But in many cases and particularly for our case, since the external excitation, i.e. lip vibration is periodic, it is appropriate to assume ω(t) as a periodic function. Let its frequency be Ω and period $T = \frac{2\pi}{\Omega}$. Hence it is possible to choose solutions of equation (1) as x_1 and x_2, the transformation of which under $t \rightarrow t+T$ will lead to general form

$$x_1(t) = \alpha^{\frac{t}{T}}\phi_1(t)$$
$$x_2(t) = \alpha^{-\frac{t}{T}}\phi_2(t) \qquad (6)$$

with positive and negative real values of constant $\alpha(|\alpha| \neq 1)$ where ϕ(t)'s are purely periodic functions of time with period T. This causes, the system instable and as it grows, one of the solutions grows rapidly, which is onset of the resonance.

In order to determine the condition of resonance let us explicitly assume ω(t), as

$$\omega^2(t) = \omega_0^2(1 + h\cos\Omega t). \tag{7}$$

Here h is a parameter physically related to the driving amplitude of vibration. On insertion of this form of ω(t), the oscillator equation (1) reads,

$$\ddot{x} + \omega_0^2[1 + h\cos\Omega t]x = 0. \tag{8}$$

A general form of the equation is

$$\ddot{x} + \omega_0^2(a + 2q\cos\Omega t)x = 0 \tag{9}$$

which is well known as Mathieu's equation, with $a = \left(\frac{2\omega_0}{\Omega}\right)$.

The solutions of Mathieu's equation are "Mathieu's functions". They can be expressed only as infinite series. Using numerical integration of equation (9), the boundary lines between stable and unstable solutions are plotted (figure 2).

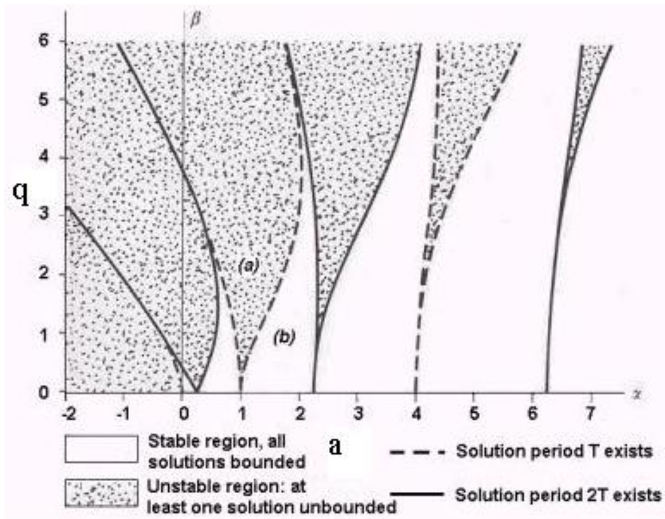

Fig. 2. Standard Plot of regions of instability of Mathieu's functions.

It may be noticed that for $\Omega = 2\omega_0$, (a =1) there is no stable region at all. An arbitrarily small driving amplitude q will drive the system to parametric resonance. Hence parametric resonance is strongest if the frequency of the function $\omega(t)$ is nearly twice ω_0. Hence, let us put, $\Omega = 2\omega_0 + \varepsilon,$ where $\varepsilon \langle\langle \omega_0$.

Therefore the oscillator equation (9) goes to

$$\ddot{x} + \omega_0^2 \left[1 + h\cos\left(2\omega_0 + \varepsilon\right)t\right]x = 0 \tag{10}$$

Its solution has been sought in the form

$$x(t) = a(t)\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t + b(t)\sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t \tag{11}$$

Hence resonant mode with natural frequency of ω_0 is the simplest possibility. Of course, this is a case of parametric resonance where the amplitude at the natural frequency is amplified. The case of pumping of a swing seems to confirm as manifestation of this solution. However, Landau & Lifshitz clearly state that the solution in equation(11) is not exact. The solution *x(t)* involves terms with frequencies which differ from $\omega_0 + \frac{1}{2}\varepsilon$ by integral multiples of $\left(2\omega_0 + \varepsilon\right)$, i.e. sine and cosine terms with arguments $\left(\omega_0 + \frac{1}{2}\varepsilon\right) + n\left(2\omega_0 + t\right)$, with *n=1, 2, 3….* But such terms involve coefficients as we will see below, which are of higher order in the parameter *h*; and hence they are, in the instances of mechanics, neglected. Not only that, even in the case of pendulum, whose point of support oscillates vertically is pointed out not to oscillate; it rather is set to spin.

Here we identify the presence of the term with frequencies around $3\omega_0$, $5\omega_0$ etc. as the signals of onset of mode locking in acoustic systems. We may go further to assume, instead of remaining fixed at very small value *h*, the driving amplitude, in lip blowing may be increased by increasing the blowing pressure. Hence, the higher order terms succeed and predominate, than remaining negligible. We would also see in the following that the process is further assisted by the presence of damping in the oscillating air column, which would explain the requirement of threshold amplitude, and possibly also that of stability of mode-locking.

4. Derivation of Range of Instability

We present in the following the outlines of calculations and results of range of instability (or regions of resonance) for solutions in successive approximation, as per the elegant method of Landau & Lifshitz. In the first case the substitution of the solution given by equation (11) in equation (10) provides the following two differential equations in *a(t)* and *b(t)*, up to first order in ε;

$$\begin{aligned} 2\dot{a} + b\varepsilon + \frac{1}{2}h\omega_0 b = 0 \\ 2\dot{b} - b\varepsilon + \frac{1}{2}h\omega_0 a = 0 \end{aligned} \tag{12}$$

If we desire solutions for a and b in the form $\exp(gt)$, to cause resonance effect, on use of such form we arrive at the equations,

$$\begin{aligned} ga + \frac{1}{2}\left(\varepsilon + \frac{1}{2}h\omega_0\right)b = 0 \\ \frac{1}{2}\left(\varepsilon - \frac{1}{2}h\omega_0\right)a - gb = 0 \end{aligned} \tag{13}$$

Now the compatibility conditions turn out to be

$$g^2 = \frac{1}{4}\left[\left(\frac{1}{2}h\omega_0\right)^2 - \varepsilon^2\right]. \tag{14}$$

The requirement for on-set of instability is that, g must be real, i.e. $g^2 \rangle 0$, which implies that the range of instability is

$$-\frac{1}{2}h\omega_0 \langle \varepsilon \langle \frac{1}{2}h\omega_0 . \tag{15}$$

The same process can be repeated on inclusion of the terms with frequency around $3\omega_0$, in the solution (11), for which the solution will read as;

$$x = a_0 \cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t + b_0 \sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t + a_1 \cos 3\left(\omega_0 + \frac{1}{2}\varepsilon\right)t + b_1 \sin 3\left(\omega_0 + \frac{1}{2}\varepsilon\right)t \quad . \tag{16}$$

In order to obtain the limit of instability (range of resonance) the coefficients *a* and *b* may be treated as constants here.

Substituting solution (16) into equation (10) and retaining terms involving factors upto ε^2, we get,

$$\left[-a_0\left(\omega_0\varepsilon+\frac{1}{4}\varepsilon^2\right)+\frac{1}{2}h\omega_0^2a_0+\frac{1}{2}h\omega_0^2a_1\right]\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t+$$
$$\left[-b_0\left(\omega_0\varepsilon+\frac{1}{4}\varepsilon^2\right)-\frac{1}{2}h\omega_0^2b_0+\frac{1}{2}h\omega_0^2b_1\right]\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t+$$
$$\left[\frac{1}{2}h\omega_0^2a_0-8\omega_0^2a_1\right]\cos 3\left(\omega_0+\frac{1}{2}\varepsilon\right)t+$$
$$\left[\frac{1}{2}h\omega_0^2b_0-8\omega_0^2b_1\right]\sin 3\left(\omega_0+\frac{1}{2}\varepsilon\right)t=0. \quad (17)$$

This equation demands that the coefficient of sine and cosine functions in each term separately vanishes. The last two terms give,

$$a_1=\frac{ha_0}{16}$$
$$b_1=\frac{hb_0}{16}. \quad (18)$$

And the first two equations lead to the pair of relations given by the equations.

$$\omega_0\varepsilon\pm\frac{1}{2}h\omega_0^2+\frac{1}{4}\varepsilon^2-\frac{h^2\omega_0^2}{32}=0, \quad (19)$$

which has the solution $\varepsilon=\pm\frac{h\omega_0}{2}-\frac{h_0^2\omega_0}{32}.$ (20)

In the next step let us include *cos* $5\omega_0 t$ term in the solution, so that it will read

$$\begin{aligned} x= & a_0 \cos\left(\omega_o+\frac{\varepsilon}{2}\right)t+b_0 \sin\left(\omega_o+\frac{\varepsilon}{2}\right)t \\ & +a_1 \cos 3\left(\omega_o+\frac{\varepsilon}{2}\right)t+b_1 \sin 3\left(\omega_o+\frac{\varepsilon}{2}\right)t \\ & +a_2 \cos 5\left(\omega_o+\frac{\varepsilon}{2}\right)t+b_2 \sin 5\left(\omega_o+\frac{\varepsilon}{2}\right)t. \end{aligned} \tag{21}$$

Upon use of this solution in equation (10) and retaining the terms involving trigonometric functions with argument up to $5\omega_0 t$, one arrives at an equation as a sum of six terms involving sine and cosine functions of different arguments. And validity of the equation therefore demands that; the coefficient of each periodic function separately must vanish. This leads to the following six equations, namely,

$$\omega_0^2 a_0-\left(\omega_0+\frac{\varepsilon}{2}\right)^2 a_0+\frac{h a_0 \omega_0^2}{2}+\frac{h\omega_0^2 a_1}{2}=0 ....(a)$$

$$\omega_0^2 b_0-\left(\omega_0+\frac{\varepsilon}{2}\right)^2 b_0-\frac{h b_0 \omega_0^2}{2}+\frac{h\omega_0^2 b_1}{2}=0 ....(b)$$

$$\omega_0^2 a_1-9a_1\left(\omega_0+\frac{\varepsilon}{2}\right)^2+\frac{h a_0 \omega_0^2}{2}+\frac{h\omega_0^2 a_2}{2}=0 ....(c)$$

$$\omega_0^2 b_1-9b_1\left(\omega_0+\frac{\varepsilon}{2}\right)^2+\frac{h b_0 \omega_0^2}{2}+\frac{h\omega_0^2 b_2}{2}=0 ....(d)$$

$$\omega_0^2 a_2-25a_2\left(\omega_0+\frac{\varepsilon}{2}\right)^2+\frac{h a_1 \omega_0^2}{2}=0 ....(e)$$

$$\omega_0^2 b_2-25b_2\left(\omega_0+\frac{\varepsilon}{2}\right)^2+\frac{h b_1 \omega_0^2}{2}=0 ....(f)$$

(22)

From equation (22e) and (22f), we get $a_2=\frac{a_1 h}{48}$ and $b_2=\frac{b_1 h}{48}$.

Neglecting ε^2 and $\omega_0\varepsilon$ in comparison to ω_0^2; similarly from eqn.(22c) and equation (22d), we get,

$$b_1 = \frac{b_0 h}{2} \times \frac{1}{\left(8 - \frac{h^2}{96}\right)} = \frac{b_0 h}{16}\left(1 + \frac{h^2}{96 \times 8}\right)$$

and

$$a_1 = \frac{a_0 h}{2} \times \frac{1}{\left(8 - \frac{h^2}{96}\right)} = \frac{a_0 h}{16}\left(1 + \frac{h^2}{96 \times 8}\right).$$

From equation (22a) and equation (22b), consistency condition demands that

$$\omega_0 \varepsilon \pm \frac{h\omega_0^2}{2} + \frac{\varepsilon^2}{4} + \frac{h^2 \omega_0^2}{32}\left(1 + \frac{h^2}{96 \times 8}\right) = 0.$$

Since we intend to retain terms up to h^3, only,

$$\omega_0 \varepsilon \pm \frac{h\omega_0^2}{2} + \frac{\varepsilon^2}{4} + \frac{h^2 \omega_0^2}{32} = 0. \tag{23}$$

The solution of this equation for ε gives,

$$\varepsilon = \omega_0 \left[\pm \frac{h}{2} - \frac{h^2}{32} \pm \frac{3}{128} h^3 \right]. \tag{24}$$

The ranges of instability for the presence of different modes of parametric vibrations are clearly indicated in the equation above.

The solution says that resonant modes with frequencies ω_0, and its odd multiples do simultaneously exist, with the range of resonance specified by equation (24).

However, locking in takes places at different mode, as we would see below, in presence of damping calling for definite, threshold driving amplitude.

5. Damping and Threshold Driving Amplitude

As pointed out in section (3), the modes of vibrations of a stretched string in Melde's experiment, with periodic variation of tensile strength due to action of external agency was first studied by Rayleigh [11] and subsequently by Raman[12] and both the authors used Mathieu's equation for their analysis. However, these early pioneers did not designate the problems as a case of parametric resonance. Nevertheless, they arrived upon a number of important conclusions out of their analysis.

It may be noted from the form of solution given by equation (6), the motion that results from parametric excitation is unstable and grows exponentially with time. This aspect has been emphasized by Rowland [7] through graphical demonstration, of course in their case, in presence of non-linearity. This author clearly states that damping results in the solution eventually approaching a steady state. Of course, this was predicted by Raman[12] as pointed out by Rowland[7]. Both Rayleigh and Raman used a modified Mathieu's equation of the form, (in our notation)

$$\ddot{x} + \lambda\dot{x} + \omega_0^2\left(1 + h\cos\Omega t\right)x = 0. \tag{25}$$

The steady state conditions which they obtain are identical with our results for range of instability with an additional damping term.

Damping introduces a further consequence with greater relevance for the present context of mode-locking. It is a matter of observation that mode-locking at definite higher frequencies occurs on blowing the pipe or conch shell at increasing blowing pressure, i.e. blowing amplitude subject to a certain minimum (threshold), characteristic of each case. An explanation of this requirement seems to peep in Raman's 1912 work [12]. Raman states, (at page -28 of the paper) that remarkable changes are observed when the tension was smaller still. The damping was large and a steady motion was possible when the amplitude exceeded a certain minimum value. Of course, Raman takes note of such observations while dealing with theof vibration of strings. But the phenomenon becomes pronounced as a daily experience in the lips of the player blowing pipe, conical horn or conch shell. It is most distinctly felt in blowing of the Kalingan horn (Kāhāli) and conch shell.

A precise, but elegant explanation of the phenomenon is found in Landau et.al.'s classic text[1].There authors mention that parametric resonance is maintained in presence of a slight friction, with a rather narrower region of instability. In fact friction introduces a damping of amplitude of oscillation as $e^{-\lambda t}$. Hence the amplification of oscillations in parametric resonance is as $\exp[(g-\lambda)t]$. And the instability is to be decided by the condition, $g - \lambda = 0$ Therefore, one gets, in the lowest approximation,

$$-\sqrt{\left[\left(\frac{1}{2}h\omega_0\right)^2-4\lambda^2\right]}\langle\varepsilon\langle\sqrt{\left[\left(\frac{1}{2}h\omega_0\right)^2-4\lambda^2\right]} \qquad (26)$$

It is evident from equation (26) that resonance is possible not for any arbitrarily small driving amplitude *h*, but only for *h*, exceeding a threshold value h_0, given by

$$h_0=\frac{4\lambda}{\omega_0} \qquad (27)$$

Only $\cos\left(\omega_0+\frac{\varepsilon}{2}\right)t$ and $\sin\left(\omega_0+\frac{\varepsilon}{2}\right)t$ oscillators are excited with this threshold. Similar expression can be derived for higher approximations holding the excitation (locking-in) of modes with $\left(3\omega_0+\frac{\varepsilon}{2}\right)t$ and $\left(5\omega_0+\frac{\varepsilon}{2}\right)t$ frequencies.

The origin of approach is the time variation of the coefficients, and hence the differential equations in *a* and *b* as given in equation (12). But equation (12) stands only for approximation, allowing only $\cos\left(\omega_0+\frac{\varepsilon}{2}\right)t$ and $\text{sine}\left(\omega_0+\frac{\varepsilon}{2}\right)t$ in the solution. Upon including $\cos 3\left(\omega_0+\frac{\varepsilon}{2}\right)t$ and its sine companion; and also $\cos 5\left(\omega_0+\frac{\varepsilon}{2}\right)t$ and its companion term, given by equation (20), one confronts at least three pairs of amplitudes (a_0, b_0), (a_1, b_1) and (a_2, b_2). Of course, in principle, all of them are time varying. But we should note that, the coefficients a_1, a_2, and b_1, b_2 are finally rendered in terms of a_0 and b_0 respectively. Further, they are of the order of *h* and h^2. So, to avoid complicacy of calculation, not sacrificing, the seminal results, we consider up to the first derivative of the time variation of a_0 and b_0 only. As a result, equation (22a) and equation (22b) modified with one additional term of $\dot{a}_0$ and $\dot{b}_0$ respectively; read

$$-2\dot{a}_0\omega_0 - b_0\left(\omega_0+\frac{\varepsilon}{2}\right)^2 + \omega_0^2 b_0 - \frac{\omega_0^2 b_0 h}{2} + \frac{\omega_0^2 b_1 h}{2} = 0$$

and . (28)

$$2\dot{b}_0\omega_0 - a_0\left(\omega_0+\frac{\varepsilon}{2}\right)^2 + \omega_0^2 a_0 + \frac{\omega_0^2 a_0 h}{2} + \frac{\omega_0^2 a_1 h}{2} = 0$$

They further simplify, with $a(t) \approx exp(gt)$,to

$$-2ga_0\omega_0 - b_0\omega_0\varepsilon - \frac{b_0\varepsilon^2}{4} + \frac{h\omega_0^2 b_0}{2} - \frac{h^2\omega_0^2 b_0}{32} = 0$$

and (29)

$$2gb_0\omega_0 - a_0\omega_0\varepsilon - \frac{a_0\varepsilon^2}{4} - \frac{h\omega_0^2 a_0}{2} - \frac{h^2\omega_0^2 a_0}{32} = 0$$

In presence of damping, as mentioned earlier, g is to be replaced by $(g-\lambda)$ and condition of instability $g-\lambda=0$ or $g=\lambda$ be imposed. Upon substitution of these conditions, equation (28) is recast as,

$$2\lambda a_0 = \frac{h\omega_0}{2} - \left(\frac{h^2\omega_0}{32} + \varepsilon\right)$$

$$2\lambda b_0 = \frac{h\omega_0}{2} + \left(\frac{h^2\omega_0}{32} + \varepsilon\right). \tag{30}$$

Now the compatibility condition demands that

$$4\lambda^2 = \frac{h^2\omega_0^2}{4} - \left(\frac{h^2\omega_0}{32} + \varepsilon^2\right)^2 \tag{31}$$

or
$$\left(\frac{h^2\omega_0}{32} + \varepsilon^2\right) = \pm\sqrt{\left(\frac{h\omega_0}{2}\right)^2 - 4\lambda^2}. \tag{32}$$

Hence the range of instability is given by

$$\left(-\sqrt{\left(\frac{h\omega_0}{2}\right)^2 - 4\lambda^2} - \frac{h^2\omega_0}{32}\right)\langle\varepsilon\langle\left(\sqrt{\left(\frac{h\omega_0}{2}\right)^2 - 4\lambda^2} - \frac{h^2\omega_0}{32}\right) \tag{33}$$

We have to resort to limit *ε=0*, for deciding the threshold. Using it in equation (31), we obtain,

$$\frac{h^4\omega_0^2}{(32)^2} - \frac{h^2\omega_0^2}{4} + 4\lambda^2 = 0 \tag{34}$$

This may be rewritten as,

$$\frac{h^4}{(32)^2} - \frac{8h^2}{32} + \frac{4\lambda^2}{\omega_0^2} = 0 \tag{35}$$

When solved for h^2,

$$\frac{h^2}{16} = 8 \pm 8\sqrt{1 - \frac{\lambda^2}{4\omega_0^2}} \tag{36}$$

Now expanding the quantity under radical sign as a binomial series we have,

$$\frac{h^2}{16} = \left[\frac{\lambda^2}{\omega_0^2} + \frac{\lambda^4}{16\omega_0^4} + \frac{\lambda^6}{128\omega_0^6} + ...\right] \tag{37}$$

One may check that in the lowest approximation, retaining only the term in λ^2 one gets, $h_0 = \frac{4\lambda}{\omega_0}$, as decided by Landau et.al.

Equation (36) further shows that the threshold amplitude of the driving force gradually increases on inclusion of higher order terms in λ i.e. for excitation of higher modes in the 'locking-in' process.

The formalism discussed in this paper further demonstrates that the "lock - in" frequency may slightly vary from the exact odd multiples of the natural frequency by the difference $\frac{n\varepsilon}{2}$. Our experimental analysis of spectrum for both cylindrical pipes and conch- shell supports this theoretical model. It is worth pointing out here that Fletcher[16] comes to a similar conclusion, while giving the "lock-in" frequency as

$$f_i = i\omega + \Delta_i \tag{38}$$

where f_i is the i^{th} locked-in frequency, ω, is the natural frequency and Δ_i , is a small difference. However, equation (38) given by Fletcher fails to explain why only the odd multiple frequencies appear in the mode-locking as experimentally observed by us.

6. Conclusion

'Parametric resonance' and 'mode-locking' are specific phenomena dealt in mechanics and acoustics respectively. But they do also occur in fields beyond their original identity, as indicated in the introduction. We have made an attempt in this work to use the theory of parametric resonance to explain the phenomenon of mode-locking in acoustics, particularly that appearing in blowing of pipes and conch shell; the details of spectral characteristic of which have been reported recently[13]. In this paper we have presented only the sample spectrum of our observation along with the data table for a lip-driven tube. Our formalism proves effective in explaining not only the mode-locking at odd harmonics of the fundamental for systems under test, but also the existence of threshold driving amplitude. This is a simpler and more quantitative, as well as complete explanation than the earlier ones put forth by Fletcher, which have been briefly touched in the text.

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