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Analysis of some parameters associated with binary black hole merger

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Abstract. Two black holes orbiting each other due to gravitational attraction can setup a binary system. Coalescence of two black holes of a binary system creates disturbance in space-time curvature which produces gravitational waves. The first experimental observation of gravitational wave signal from a binary black hole merger by the coalescence of two black holes was made by LIGO detectors at Livingston and Hanford on 14th Sept, 2015 which ensured the physical existence of black holes and binary black hole systems. In this paper, we review and estimate theoretically the angular momentum, chirp mass, energy, power of the coalescing binary systems, mass of the newly formed black hole and peak frequency of gravitational waves emitted during inspiral, merger and ring down phases for the recently observed four binary black hole merging events GW150914, GW151226, GW170104 and GW170814 by some lucid mathematical calculations.

Keywords. General theory of relativity; black hole; neutron star; gravitational wave

1. Introduction

Gravitational wave is the consequence of *General Theory of Relativity* (GTR) proposed by Einstein over a century ago. Einstein's field equations are nonlinear in nature as gravitational field itself carries energy and momentum. Einstein's field equation for vacuum $R\mu\nu = 0$ gives wave like solution in analogous with the wave like solution of Maxwell's equations in classical theory. Schwarzschild's solution of Einstein's field equation of gravitation,

$$d\tau^2 = [1-\frac{2MG}{r}]dt^2 - [1 - \frac{2MG}{r}]^{-1}dr^2 - r^2 d\theta - r^2 sin^2\theta d\emptyset^2 \quad (1)$$

becomes singular at radius r = 2MG (in standard coordinates) [1]. This radius 2MG is called Schwarzschild's radius and Schwarzschild's solution is valid up to this radius. Any massive body having radius less than 2MG can't be exist according to Schwarzschild's singularity. No stable massive body contains Schwarzschild's singularity but there is a possibility of very massive body to collapse to a radius less than Schwarzschild's radius. Collapse of unstable neutron stars into black holes is based on this possibility which is nothing but the singularity described by Schwarzschild. Later Oppenheimer theoretically proved the existence of black holes. This was the interpretation of non-rotating black holes. The word 'black' represents the property of light capturing inside the 'event horizon' and 'hole' corresponds to singularity of space-time curvature. As no light can escape from black holes, it is impossible to observe the black holes by using electromagnetic wave as an astronomical probe. Gravitational wave is the only option to observe the black holes as it is the disturbance or ripples of space-time curvature due to massive bodies. Gravitational waves can be produced by any massive object but that magnitude becomes so small that it isn't measurable by LIGO or any other observatories [2-6]. Coalescence of massive astronomical binary objects can only produce measurable gravitational waves.

The gravitational waves were detected on 14 September 2015 by both of the twin Laser Interferometer Gravitational-Wave Observatory (LIGO) detectors, located in Livingston, Louisiana and Hanford, Washington, simultaneously from a binary black hole merger. This is the first direct detection of gravitational waves. This not only confirms the existence of gravitational waves, a major prediction of Einstein's general theory of gravity, but it also marks the very birth of gravitational wave astronomy. 2017 Nobel Prize in Physics was awarded to Kip Thorne, Barry Barish and Reiner Weiss for their *decisive contribution to the LIGO detector and observations of gravitational waves*.

Most of the gravitational wave events detected by LIGO till now are originated from binary black hole mergers; these are GW150914, GW151226, GW170104, GW170608 and GW170814. The rotating primary and secondary black holes have their own mass, energy and angular momentum. We can find out energy, power and momentum carried by the gravitational radiation due to binary coalescence theoretically from the energy-momentum tensor. Due to lack of data of the event GW170608, we analyse the rest four events of binary black hole mergers. We use initial data like mass of primary and secondary black holes of the binary, distance between two black holes, luminosity, frequency of gravitational wave spectrum etc. from LIGO factsheet.

2.1 Mass of newly formed Kerr black hole

In the approximation of particle absorption, mass of final black hole varies with mass and momentum of the particle plunged into the primary black hole. Usually final black hole forms with mass less than the primary black hole. This is called irreducible mass. Rest of the mass i.e. reducible mass [7] of primary black hole converts to energy which is carried by emitted gravitational waves. To estimate the mass of newly formed black hole, we have used the following expression which was derived from energy conservation principle [8]:

$$M_{BH} \approx M\text{-}(1\text{-}2\sqrt{2})\nu \text{ - } \Delta E_{merger} /c^2 \tag{2}$$

$$\approx M\beta(\nu), \tag{3}$$

where $\beta(\nu) \equiv [1\text{-}(1\text{-}2\sqrt{2}/3)\nu - 0.24\nu^2]$ and $\nu = \frac{\mu}{M}$ is so called symmetric mass-ratio parameter, with $\mu = \frac{m_1 m_2}{M}$ is binary reduced mass and M is total binary mass i.e. (m_1+m_2). m_1 and m_2 are two masses of the binary system. ΔE_{merger} is the energy released during merger phase. Here, all the masses are expressed in terms of solar mass i.e. M_Θ ($M_\Theta = 1.99\times10^{30}$ Kg).

Table 1. Mass of newly formed black hole

GW Event	**Binary Mass $(\mathbf{m_1, m_2})$ ($\mathbf{M_\Theta}$)**	**$\mathbf{M_{BH}}$ (our result) ($\mathbf{M_\Theta}$)**	**$\mathbf{M_{BH}}$ (LIGO data) ($\mathbf{M_\Theta}$)**
GW150914	~ (36, 29)	63.129	62 [9]
GW151226	~ (14, 7)	20.485	21 [10]
GW170104	~ (31, 19)	48.660	49 [11]
GW170814	~ (31, 25)	54.380	53 [12]

2.2 Angular momentum

Binary black hole systems have both spin and orbital angular momentum. Both primary and secondary black holes have spin angular momentum and in between them orbital angular momentum exists. In order to estimate the total angular momentum of a newly formed black hole we used the following formula based on angular momentum conservation [8]

$$J_{BH} = J_{LSO} - \Delta J_{merger} , \tag{4}$$

where JLSO is total angular momentum of inspiral phase and ΔJmerger is loss of total angular momentum of the binary system. So, the spin angular momentum of the newly formed black hole can be written as

$$\alpha \equiv \frac{cJ_{BH}}{GM_{BH}^2} \approx \frac{2\sqrt{3}\nu - 3.81\nu^2}{\beta(\nu)^2} \;, \tag{5}$$

where $\beta(\nu)$ and ν are the same as defined before and G = 6.67408x10^{-11} m^3-Kg^{-1}-sec^{-2} is the gravitational constant.

Table 2. Spin angular momentum of the newly formed black hole

GW Event	**Binary Mass** $(\mathbf{m_1, m_2})$ **($M_\odot$)**	$\boldsymbol{\alpha}$**(our result)**	$\boldsymbol{\alpha}$**(LIGO data)**
GW150914	~ (36, 29)	0.660	0.67 [9]
GW151226	~ (14, 7)	0.611	0.74 [10]
GW170104	~ (31, 19)	0.638	-
GW170814	~ (31, 25)	0.661	-

2.3 Chirp mass of binary system

In regime one i.e. within point like approximation, chirp mass of the binary can be determined as [8]

$$M_{chirp} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} \,, \tag{6}$$

where m_1 and m_2 are mass of primary black hole and test-particle respectively. Experimentally from frequency and strain data of emitted gravitational wave, chirp mass can be measured accurately. Theoretically from binary chirp mass we can measure the frequency evolution of the resultant gravitational wave as [6]:

$$M_{chirp} = \frac{c^3}{G}\left(\frac{5}{96\pi^{8/3}}\frac{\dot{f}}{f^{11/3}}\right)^{3/5} , \tag{7}$$

where f is frequency of gravitational wave, $\dot{f}$ is the change of frequency with respect to time.

Table 3. Chirp mass of binary system

GW Event	**Binary Mass (m_1, m_2) ($M_\odot$)**	**Mchirp (our result) (MΘ)**	**Mchirp (from LIGO data) (MΘ)**
GW150914	~ (36, 29)	28.09	30.2 [9]
GW151226	~ (14, 7)	8.52	8.9 [10]
GW170104	~ (31, 19)	21.00	21.1 [11]
GW170814	~ (31, 25)	24.20	-

2.4 Energy carried by gravitational wave

According to classical theory of quadrupole radiation in point-like approximation, rate of loss of energy with frequency of the binary due to the absorption of the test-particle [8]

$$\frac{dE}{df} = \frac{1}{3}(\pi G)^{2/3} \nu M^{5/3} f^{-1/3} , \tag{8}$$

where $\nu = \frac{\mu}{M}$ is symmetric mass ratio in which μ is reduced mass, M is total binary mass, G is gravitational constant and f is gravitational wave frequency. From the gravitational wave spectrum we can write an approximate working formula of the spectrum as

$$\frac{dE}{df} \approx \left[\frac{1}{\frac{dE}{df}_{plunge}} + \frac{1}{\frac{dE}{df}_{ringdown}}\right]^{-1} , \tag{9}$$

Total energy carried by gravitational wave is

$$\Delta E_{merger} = \sum_{l \geq 2} \int df \, \frac{dE}{df}_{2^l - pole} \tag{10}$$

$$\approx 0.01 \frac{\mu^2}{M} c^2 , \tag{11}$$

where l is the order of pole and μ is reduced mass, M is total binary mass and c is speed of light in vacuum.

Table 4. Energy carried by gravitational wave

GW Event	Binary Mass ($\mathbf{m_1, m_2}$) ($\mathbf{M_\Theta}$)	ΔEmerger (our result) ($\mathbf{M_\Theta}$ c2)	ΔEmerger (our result) ($\mathbf{M_\Theta\ c^2}$)
GW150914	~ (36, 29)	3.96	2.5-3.5 [9]
GW151226	~ (14, 7)	1.04	0.8-1.1 [10]
GW170104	~ (31, 19)	2.70	1.3-2.6 [11]
GW170814	~ (31, 25)	3.42	2.4-3.1 [12]

2.5 POWER CARRIED BY GRAVITATIONAL WAVE

We have used the general mathematical formula for a binary system to calculate the power carried by gravitational wave [13]

$$P = \frac{3}{5}\frac{G^4}{c^5}\frac{m_1^2 m_2^2 (m_1+m_2)}{R^5}, \tag{12}$$

where G is gravitational wave constant, c is speed of light in vacuum ($3\text{x}10^8$ m-sec^{-1}), m_1 and m_2 are component masses of the binary and R is the intermediate distance between the two sources.

Table 5. Power carried by gravitational wave

GW Event	Binary Mass ($\mathbf{m_1, m_2}$) ($\mathbf{M_\Theta}$)	Orbital separation R (meter)	Power P (our result) (watt)	Peak luminosity (LIGO data) (watt)
GW150914	~ (36, 29)	$6.43\text{x}10^5$	$1.04\text{x}10^{49}$	$3.6\text{x}10^{49}$ [9]
GW151226	~ (14, 7)	$2.14\text{x}10^5$	$7.25\text{x}10^{47}$	$2\text{x}10^{49} – 4\text{x}10^{49}$ [10]
GW170104	~ (31, 19)	$5.23\text{x}10^5$	$7.21\text{x}10^{47}$	$1.8\text{x}10^{49} – 3.8\text{x}10^{49}$ [11]
GW170814	~ (31, 25)	$5.26\text{x}10^5$	$1.36\text{x}10^{48}$	$3.2\text{x}10^{49} – 4.2\text{x}10^{49}$ [12]

2.6 Peak frequency of gravitational wave spectrum

Gravitational wave spectrum is dominated by quadrupole emission and that the largest gravitational wave emission occurs from r ≈ 3GM/c^2, at the maximum of the effective potential [8]

$$V_l\,(r) = (1- \frac{2m_{BH}}{r})[\frac{2\lambda^2(\lambda+1)r^3+6\lambda^2 m_{BH}r^2+18\lambda m_{BH}^2 r+18m_{BH}^3}{r^3(\lambda r+3m_{BH})^2}], \tag{13}$$

where λ = $(l - 1)(l + 2)/2$ and l is the no. of poles. In the limit of large l, the contribution of each multipole (quadrupole) to the spectrum peaks at the gravitational wave frequency

$$f_{peak}^{l} = \frac{c^3}{G}\sqrt{(V_l)_{\max}} \tag{14}$$

$$\approx \frac{c^3}{G}\frac{1}{2\pi\sqrt{27M}}\,, \tag{15}$$

The total spectrum can be obtained by summing over all the multipoles

$$f_{peak} \approx \frac{c^3}{G}\frac{0.05}{M} \tag{16}$$

$$\approx 10.36\frac{M_\odot}{M}\ \text{KHz}\,, \tag{17}$$

where MΘ is the mass of sun (~ 1.99×10^{30} kg) and M is total binary mass.

Table 6. Peak frequency of gravitational wave spectrum

GW Event	**Binary Mass ($\mathbf{m_1, m_2}$) ($\mathbf{M_\odot}$)**	**$\mathbf{f_{peak}}$ (our result) (Hz)**	**$\mathbf{f_{peak}}$ (from LIGO data) (Hz)**
GW150914	~ (36, 29)	159.38	150 [9]
GW151226	~ (14, 7)	493.30	420 [10]
GW170104	~ (31, 19)	207.20	160 – 199 [11]
GW170814	~ (31, 25)	185.00	155 – 203 [12]

3. Concluding Remarks

The existence of gravitational waves [13-16] is one of the most intriguing predictions of Einstein's GTR. In this paper, we review and estimate the angular momentum, chirp mass, energy, power of the coalescing binary systems, mass of the newly formed black hole and peak frequency of gravitational waves emitted during inspiral, merger and ring down phases for the recently observed four binary black hole merging events GW150914, GW151226, GW170104 and GW170814 using simple mathematical calculations. We have found that in each case our estimated values for each parameter and for each event are well agreed with experimental data observed by LIGO team.

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