Orissa Journal of Physics
ISSN 0974-8202

Vol. 22, No.2
August 2015
pp. 255-258

Application of Hyperreal Numbers in Physics and Related Fields

P. K. KARN

Department of Mathematics, MMAM Campus, Biratnagar
Tribhuvan University, Nepal

Received : *2.6.2015* ; *Accepted : 14.07.2015*

Abstract: In this paper we have pointed out with a philosophical discussion that the nonstandard elements have not only wide scope of application in physics but also in other related fields e.g. mathematics, statistics etc.

1. Introduction

A thorough and detailed application of the nonstandard elements has already been made by Robinson whose approach is based on logic [1]. However, it will really be useful if one makes their application without any appeal to logic. We give here a brief overview of some of them.

2. Hyperreal Numbers

Hyperreal numbers can help us to understand why results that are obtained in a non-rigorous way are nevertheless correct. For the sake of illustration and brevity we focus on a single notion from the calculus - the derivative. The derivative expresses the rate of change of the dependent variable with respect to the change in the independent one. If the independent variable is interpreted as time and the dependent variable as displacement, the derivative is the (instantaneous) velocity. If the independent variable is plotted as abscissa and the dependent variable as ordinate, the derivative is the slope of the tangent at a point on the graph of the function.

Leibnitz’s development of the calculus (around 1674) is characterised by three important ingredients:

(i) Use of infinitesimals: Leibnitz’s notation of the derivative as a quotient of infinitesimals e.g. dy/dx is still in use in mathematics today although in standard analysis the derivative cannot be interpreted as a quotient.

(ii) Law of continuity: The law of continuity or Souverian principle says that the rules of the finite also hold in the infinite and vice versa. It is related to the transfer principle in nonstandard analysis (NSA).

(iii) Law of transcendental homogeneity: The law of transcendental homogeneity says that when comparing two quantities, the quantity with a lower order of infinity can be ignored. It is related to the standard part function in NSA [3].

Newton developed his version of the calculus (around 1666) with physics in mind which led him to a dynamic concept of the derivative. He thought of the derivative of a continuous function and called as the fluxion of a fluent. His dot notation of the derivative e.g. $\dot{x}$ for dx/dt is still in use in physics today although Newton did not base his calculus on the notion of the infinitesimal as Leibnitz did. However, infinitesimals do appear in his work too—both as infinitely small period of time and as moment of fluent quantity.

Berkeley famously criticised the use of infinitesimals and evanescent quantities in his work 'The Analyst'. Berkley was so influenced that many believed that infinitesimal had to be banned from mathematics once and for all. However, Katz and Sherry recently (2012) pointed out a flaw in Berkeley's criticism.

3. Standard and Nonstandard Analysis

The modern approach to standard analysis was developed by the great triumvirate Cantor, Dedekind and Weierstrass. For limit Weierstrass introduced the modern epsilon-delta definition which goes back to Bolzano in 1817. This allows us to define the derivative as a limit of the quotient of differences [2]:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x},$$

where $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = L \Leftrightarrow \forall \varepsilon\ (>0) \in R,\ \exists\ a\delta\ (>0) \in R,\ \forall \Delta x \in R$$

$$0 < |\Delta x| < \delta \Rightarrow \left| \frac{\Delta y}{\Delta x} - L \right| < \varepsilon$$

In nonstandard analysis the classical derivative is defined by Robinson as the standard part of a quotient of infinitesimals in R^*:

$$\frac{dy}{dx} = \mathrm{St}\ \frac{y^*(x+\delta) - y^*(x)}{\delta},$$

where $y^*(x) : R^* \to R^*$ is the hyperextension of the real function $y(x) : R \to R$ and δ is an infinitesimal in R^*.

In standard analysis the sequence $<0.1, 0.01, 0.001, \ldots, 10^{-n}, \ldots>$ converges to 0. It is a null sequence. Likewise in standard analysis 0.999… is exactly equal to (or just a different notation for) 1 for an amusing overview of various proofs. Nevertheless, the intuition that 0.999… is infinitesimally smaller than unity is a resilient one-not only among students in mathematics class every so often.

By looking at the ultrapower construction of R^* one may convince oneself that the equivalence class under a free ultrafilter of $<0.1, 0.01, 0.001, \ldots, 10^{-n}, \ldots>$ is different from that of zero. The former sequence is different from zero at every position whereas the latter sequence is exactly zero at every position. Hence, the index set of position where both sequences are exactly equal is the empty set which is not in the free ultrafilter. Therefore, they correspond to different hyperreal numbers [4].

Likewise the sequence $<0.9, 0.99, 0.999, \ldots, 1-10^{-n}, \ldots>$ is different from the constant sequence $<1, 1, 1, \ldots, 1, \ldots>$ at every position and, hence, does not belong to the same equivalence class under a free ultrafilter.

Observe that this does not contradict 0.999… = 1 since the number

$$|<0.9, 0.99, 0.999, \ldots, 1-10^{-n}, \ldots>| \approx u$$

is not equal to the real number 0.999… . In fact, by observing that the standard part of this number is 1, we could use it to prove that 0.999… = 1.

The above is a hyperreal number strictly smaller than 1, differing from it by the infinitesimal quantity

$$|<0.1, 0.01, 0.001, \ldots, 10^{-n}, \ldots>| \approx u.$$

4. Applications

4.1 In Physics

Physicists have continued to speak of infinitesimal quantities since the development of calculus, seemingly not bothered by the foundational issues that were in the mind of the mathematicians. Therefore, the combination of physics and nonstandard analysis seems to be very natural one. It allows physicists to continue their appeal to the intuitive notion of infinitesimals, now knowing that there is a rigorous mathematical basis for their concept.

Many applications of NSA in physics are related to differential equation and stochastic equation. NSA has been applied to quantum mechanics in multiple ways, including Feynman path integral and quantum field theory. Moreover, it seems to be a very natural idea to re-examine the quantum classical limit in this framework by considering Planck's constant (h) as an infinitesimal.

4.2 In Mathematics

The most important application of NSA is to make the proof shorter, easier or both-alleviating epsilon-delta management. An early expression of this can be found with Lagrange. NSA can be used to analyse common intuition concerning infinitesimals whereas the history of calculus is dominated by the concept of infinitesimals. NSA can also shed more light on problems related to infinitely large.

4.3 In Statistics

The fields of nonstandard measure theory and nonstandard probability theory are among the most developed areas of application of NSA. A probability function is regular if it only assigns probability zero to the impossible event. Because of finite additivity it is equivalent to only assigning probability one to the certain event. It is well known that standard probability functions based on Kolmogorov's axioms for probability can violate regularity in the case of countable infinite sample spaces and always do so in the uncountable case.

4.4 In Mathematical Economics

Hyperreals have been used in mathematical economics also.

5. Physics and Rethinking the Continuum

We often use continuum as a synonym for the standard reals. However, this is nothing but one formalisation of the concept of physical continuum. Hyperreals form an alternative formalisation of the concept. Like the standard reals the hyperreals are infinitely divisible. In particular, infinitesimals are infinitely divisible. However, many applications make use of hyperfinite sets, which, like a finite set, do contain a smallest non-zero element. Therefore, such models are discrete or chunky, rather than continuous. Another distinction to be made here is that, besides the hyperreals, there are other systems that also contain infinitesimals but which may have very different properties. Archimedes and Zeno, for instance, conceived of infinitesimals as dimensionless points.

6. Conclusion

Thus we see that nonstandard elements will really be useful if we make their application without any appeal to logic in different branches of physics as well as in other related fields.

References

[1] A. Robinson, *Nonstandard Analysis*, Princeton University Press, New Jersey (1996)

[2] J. Keisler, *Elementary Calculus*, University of Wisconsin, Eau Claire (2000)

[3] P. A. Loeb, *An Introduction to Nonstandard Analysis*, Kluwer Academic Publishers, Dordrecht (2000)

[4] R. Goldblatt, *Lectures on the Hyperreals: An Introduction to Nonstandard Analysis*, Springer-Verlag, Berlin (1998).