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Finite Difference Solution of the Unsteady Flow of a Third Grade Fluid Over an Infinite Flat Plate

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Abstract: Several Authors have studied the steady flow of third grade fluid past through a porous plate and obtained the solution using power series expansion and homology method. In this paper we have studied the flow of third grade fluid over an infinite flat plate which is set to motion with a time dependent velocity u(t) using an unconditionally stable finite difference method. As a result it is not necessary to restrict the parameters. The equations governing the flow are solved using damped Newton method. The effect of different flow parameters on the velocity field are discussed and reflected in the figures. The major advantage of this method lies in non-restriction of flow parameters. The results are compared with the results obtained by Rajgopal and Endogan (1995) analytically and this method gives more accurate solution.

1. Introduction

The flow of Non-Newtonian fluids is one of those areas of research which offers interesting and exciting challenges. Several studies on the flow of second order fluids have been made because of their technological importance. The second order fluid model is able to predict the normal stress differences, which are characteristics of non Newtonian fluids. However it does not exhibit the shear thinning & thickening phenomenon observed in many fluids by Jose et. al (1973), Beavers & Joseph (1975).

The third grade model attempts to include such characteristics of viscoelastic fluids. Fosdik & Raj Gopal (1980) have studied the thermodynamics & stability of fluids of third grade. Raj Gopal & NA (1983) worked on stokes problem on the flow due to an oscillating plate for fluid of grade three. Siddiqui & Kaloni (1987) worked on plane steady flows of such fluids. Erdogan (1995) considered the flow of a third grade fluid in the vicinity of a plane wall suddenly set in motion .

T.Hayat, S.Nadeem, S.Asghar & A.M. Siddiqui (2001) studied Fluctuating Flow of a third grade fluid on a porous plate in a rotating medium. In 2003 P.Donald Ariel studied flow of third grade fluid through porous flat channel & also T. Hayat, A.H.Kara, E.Momoniant found exact flow of a third grade fluid on a porous wall .

K.Fakhar, Z.C. Chen (2004) studied steady flow of third grade fluid subject to suction. Bikash Sahoo (2007) investigated the laminar flow & heat transfer of an incompressible third grade electrically conducting fluid impinging normal to a plain in the presence of a uniform magnetic field.

K.Fakhar, Z henil & Cheng Yi (2008) investigated the exact solution for an unsteady flow of an incompressible fluid of third grade occupying space over an infinite porous plate.

O Anwar Beg, S Rawat etal in 2010 researched on finite element modeling of transferring third grade viscoelastic biotechnological fluid flow in a darcian permeable half space.

Recently O.D. Makinde & T.Chinyoka (2011) researched on unsteady hydromagnetic generalized Coutte flow & heat transfer characteristics of a reactive variable viscosity incompressible electrically conducting third grade fluid in a channel with a symmetric convective cooling at the walls in the presence of uniform transverse magnetic field.

R Ellahi 2012 studied on the convergence of series solution of Non-Newtonion third grade fluid with variable viscosity. Recently (2013) Aiyesimi Y.N Okeday OG-T and Lawal O.W investigated unsteady MHD thin film flow of a third grade fluid with heat transfer and no slip boundary condition down in an incline plane.

We study the flow of a third grade fluid over an infinite flat plate which is set to motion with a time dependent velocity u(t) using finite difference method. This work includes the work of Raj Gopal (1990) & Erdogan (1995) & since an unconditionally stable finite difference method has been applied, it is not necessary to restrict the parameters.

2. Formulation of the Problem

We consider the flow near a flat plate which is suddenly moved in its own plane with a velocity u(t). The x' axis is taken along the plate in the direction of u & the y' axis perpendicular to the plate. Assuming that the side wall effects are neglected, namely the plate is infinitely long, from the equation of continuity the velocity, which has one component only can be written in the following form :

$$u' = u'(y,t))$$

The fluid is set into motion trough the action of the stress at the plate.

From the constitutive equation

$$\sigma = -pI + \sum_{i=1} S_i \qquad (1)$$

where S_i= μA_1 S_2= $\alpha_1 A_2 + \alpha_2 A_1^2$, S_3= $\beta_1 A_3 + \beta_2 (A_2 A_1 + A_1 A_2) + \beta(\mathrm{tr}\, A_2)A_1$

μ, $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$ fluid material constant, σ the stress tensor, p the pressure, I the identity tesor & A_n representing the Rivlin-Ericksen tensor. Defined by A_0 =I, $A_1 = \Delta u + (\Delta u)^T$

$A_{n+1} = (\partial/\partial t + u.\Delta)A_n + \Delta u.A_n + (\Delta u.A_n)^T$

Substituting (1) the expression for stress is given by

$$\sigma_{xx} = -p + \alpha_2 (\partial u'/\partial y')^2 + 2\,\beta_2 (\partial u'/\partial y')(\,\partial^2 u'/\partial y'\,\partial t) \qquad (2)$$

$$\sigma_{yy} = -p + (2\,\alpha_1 + \alpha_2)\,(\partial u'/\partial y')^2 + (6\,\beta_1 + 2\beta_2)\,(\,\partial u'/\partial y'\,)(\partial^2 u'/\,\partial y'\partial t') \qquad (3)$$

$$\sigma_{zz} = -p \qquad (4)$$

$$\sigma_{xy} = \mu(\partial u'/\partial y') + \alpha_1(\partial^2 u'/\partial y'\partial t') + 2\,(\beta_2 + \beta_3)\,(\partial u'/\partial y')^3 + \beta_1(\partial^3 u'/\partial y'\partial t'^2) \qquad (5)$$

$$\sigma_{xz} = 0$$

$$\sigma_{zy} = 0$$

where $\sigma_{xy} = \sigma_{yx}$, $\sigma_{xz} = \sigma_{zx}$, $\sigma_{yz} = \sigma_{zy}$

Inserting the above stress components and velocity equation (1) in the equation of motion

$\rho DV_i / Dt = -\rho\square_i + \rho X_i + P_{ij,j}$

We obtain, where V_i =velocity vector, X_i=external body force acting.

$\rho\partial u'/\partial t' = -\,\partial p/\partial x + \mu\,(\partial^2 u'/\partial y'^2) + \alpha_1(\partial^3 u'/\partial y'^2\partial t') + 6(\beta_2 + \beta_3)\,(\partial u'/\partial y')^2(\partial^2 u'/\partial y'^2) + \beta_1(\partial^4 u'/\partial y'^2 dt'^2$ (6)

$$0 = \partial p/\partial y + \partial/\partial y'\,[2\,\alpha_1 + \alpha_2)(\partial u/\partial y)^2 + (6\beta_1 + 2\beta_2)\,(\,\partial u'/\partial y')\,\partial^2 u'/\,\partial y'\,dt'] \qquad (7)$$

From equation (6) & (7) implies that $\partial p/\partial x$ depends upon 't' only. First the velocity field is formed from equation (6) & then the pressure field is obtained. Since there is no applied pressure gradient, equation (6) becomes :

$$\partial u'/\partial t' = \nu(\partial^2 u'/\partial y'^2) + \beta\,\partial^3 u'/\partial y'^2\partial t' + \gamma(\partial u'/\partial y')^2(\partial^2 u'/\partial y'^2) + \in(\partial^4 u'/\partial y'^2\partial t'^2) \qquad (8)$$

where $\nu = \mu/\rho$, $\beta = \alpha_1/\rho$, $\gamma = 6(\beta_2 + \beta_3)/\rho$, $\in = \beta_1/\rho$

The initial condition is

$t' = 0$, $u' = \partial u/\partial t = 0$ for $y' \geq 0$

the boundary conditions are

$t' > 0$: $u' = U(t')$ for $y' = 0$

$u' \to 0$ for $y' \to \infty$

For computation purpose we choose $U(t') = U_0 \cos\omega' t'$

Introducing he following non dimensional quantities

$U' = u_0'$, $y' = \nu/u_0\, y$, $t' = \nu/u_0^2\, t$

And $\beta = \bar{\beta} u_0^2/\nu^2$, $\nu = u_0^4 \nu/v^3$, $\in = \bar{\in} u_0^4/v^3$

Equation (8) reduces to,

$$\partial u/\partial t = \partial^2 u/\partial y^2 + \bar{\beta}\, \partial^3 u/\partial y^2 \partial t + \bar{\nu}\, (\partial u/\partial y)^2 (\partial^2 u/\partial y^2) + \bar{\in} (\partial^4 u/\partial y^2 \partial t^2) \qquad (9)$$

and the initial & boundary conditions reduces to

$t = 0$: $u = \partial u/\partial t = 0$, $y \geq 0$

$t > 0$: $u = U(t)$ for $y = 0$

(Where $U(t) = \cos\omega t$ for oscillating plate)

3. Solution of the Equations

Using the finite difference approximation for the derivatives.

$\partial u/\partial y = (u^{j+1}_{i+1} - u^{j+1}_{i-1})/2h$

$\partial^2 u/\partial y^2 = 1/2h^2\, [u^{j+1}_{i-1} - 2u^{j+1}_{i} + u^{j+1}_{i+1} + u^{j}_{i+1} - 2u^{j}_{i} + u^{j}_{i-1}]$

$\partial u/\partial t = (u^{j+1}_{i} - u^{j-1}_{i})/\, 2\Delta t$

$\partial^2 u/\partial t^2 = [u^{j+1}_{i} - 2u^{j}_{i} + u^{j-1}_{i}]\, /(\Delta t)^2$

$\partial^3 u/\partial y^2 \partial t = 1/2h^2\, \Delta t\ [(u^{j+1}_{i+1} - 2u^{j+i}_{i} + u^{j+1}_{i-1}) - (u^{j-1}_{i+1} - 2u^{j-1}_{i} + u^{j-1}_{i-1})]$

$\partial^4 u/\partial y^2 \partial t^2 = (u^{j+1}_{i+1} - 2u^{j+1}_{i} + u^{j+1}_{i-1}) - 2(u^{j}_{i+1} - 2u^{j}_{i} + u^{j}_{i-1})$
$\quad + (u^{j-1}_{i+1} - 2u_i^{j-1} + u^{j-1}_{i-1})/\, h^2 (\Delta t)^2$

The discretised form of equation (9)

$R_i = 1/2\Delta t\, (u^{j+1}_{i} - u^{j-1}_{i}) - 1/h^2\, [u^{j+1}_{i+1} - 2u^{j+1}_{i} + u^{j+1}_{i-1}] + \bar{\beta}/2h^2\, (\Delta t)[(u^{j+1}_{i+1} - 2u^{j+1}_{i} + u^{j+1}_{i-1}) - (u^{j-1}_{i+1} - 2u^{j-1}_{i} + u^{j-1}_{i-1})] + \bar{\nu}/4h^2 \times h^2\, [u^{j+1}_{i+1} - u^{j+1}_{i-1}]^2\, [u^{j+1}_{i+1} - 2u^{j+1}_{i} + u^{j+1}_{i-1}] + \bar{\in}/\, h^2(\Delta t^2)[(u^{j+1}_{i+1} - 2u^{j+1}_{i} + u^{j-1}_{i-1}) - 2\, [u^{j}_{i+1} - 2(u^{j}_{i} + u^{j}_{i-1}) + (u^{j-1}_{i+1} - 2u^{i-1}_{i} + u^{j-1}_{i-1})] = 0$

$i=1,2\ldots\ldots N,\ j=0,1,2\ldots\ldots M-1$ (10)

& the discretised boundary conditions are

$u^0_i = 0, i = 0,1,2\ldots\ldots N+1$

$u^{-1}_i = u^1_i, i = 0,1,2\ldots\ldots N$

$$\left.\begin{aligned} u^j_0 &= \cos(\omega j\Delta t) \\ u^j_{N+1} &= 0 \end{aligned}\right\} \quad j=1,2\ldots\ldots M \qquad (11)$$

For j=0, the equation (11) reduces to

$$R_0^{(i)} = 1/2\Delta t\,(u^1_i - u^{-1}_i) - 1/h^2\,[u^1_{i+1} - 2u^1_i + u^1_{i-1}] - \bar{\beta}/2\,h^2(\Delta t)\,[(u^1_{i+1} - 2u^1_i + u^1_{i-1}) - (u^{-1}_{i+1} - 2u^{-1}_i + u^{-1}_{i-1})] - \bar{\nu}/4h^2 x h^2\,[u^1_{i+1} - u^1_{i-1}]^2 - [u^1_{i+1} - 2u^1_i + (u^1_{i-1})] - \bar{\in}/h^2(\Delta t^2)[(u^1_{i+1} - 2u^1_i + u^1_{i-1}) - (u^{-1}_{i+1} - 2u^{-1}_i + u^{-1}_{i-1})] = 0 \qquad (12)$$

Using equation (11) in equatin (12)becomes

$$R_0(i) = 1/h^2\,(u^1_{i+1} - 2u^1_i + u^1_{i-1}) + \bar{\nu}/4h^4\,(u^1_{i+1} - u^1_{i-1})^2\,(u^1_{i+1} - 2u^i_i + u^1_{i-1}) + 2\bar{\in}/h^2(\Delta t)^2\,(u^1_{i+1} - 2u^i_i + u^1_{i-1}) = 0 \qquad (13)$$

The system of non linear equation (10), (11) & (12) with the boundary conditions of (11) are solved using Damped Newton method described in Conte De Boor (1980).

The derivatives in the Jacobian used in this method are computed as follows.

$$\partial R_0(i)/\partial u^1_{i-1} = 1/h^2 + \nu/4h^4\,(u^1_{i+1} - u^1_{i-1})^2 - 2\bar{\nu}/4h^4(u^1_{i+1} - u^1_{i-1})\,(u^1_{i+1} - 2u^1_i + u^1_{i-1}) + 2\bar{\in}/h^2(\Delta t)^2 \qquad (14)$$

$$\partial R_0(i)/\partial u^1_i = -2/h^2 - 2\bar{\nu}/4h^4\,(u^1_{i+1} - u^i_{i-1})^2 - 4\bar{\in}/h^2(\Delta t)^2 \qquad (15)$$

$$\partial R_0(i)/\partial u^1_{i+1} = 1/h^2 + \bar{\nu}/4h^4\,(u^1_{i+1} - u^i_{i-1})^2 + 2\bar{\nu}/4h^4\,(u^1_{i+1} - u^1_{i-1})\,(u^1_{i+1} - 2u\text{-}^1_i + u^1_{i-1}) + 2\bar{\in}/h^2(\Delta t)^2 \qquad (16)$$

From Equation (10) & (11) at $j=1,2\ldots\ldots M-1$ we get

$$\partial Ri/ \partial u^{J+1}_{i-1} = -/1/2\Delta t - 2/h^2 - 2\bar{\beta} /2h^2\Delta t - 2\bar{\nu} / 4h^4 (u^{j+1}_{i+1} - u^{j+1}_{i-1})^2 - 2\bar{\in} / h^2(\Delta t)^2 \quad (17)$$

$$\partial Ri/ \partial u^{J+1}_{i-1} = 1/h^2 + \bar{\beta} /2h^2\Delta t - 2\bar{\nu} / 4h^4(u^{j+1}_{i+1} - u^{j+1}_{i-1}) (u^{j+1}_{i+1} - 2u^{j+1}_{i} + u^{j+1}_{i-1}) + \bar{\nu} / 4h^4(u^{j+1}_{i+1} - u^{j+1}_{i-1}) + \bar{\in} / h^2(\Delta t)^2 \quad (18)$$

$$\partial Ri/ \partial u^{J+1}_{i+1} = 1/h^2 + \bar{\beta} / 2h^2\Delta t - 2\bar{\nu} /4h^4(u^{j+1}_{i+1} - u^{j+1}_{i-1}) (u^{j+1}_{i+1} - 2u^{j+1}_{i} + u^{j+1}_{i-1}) + 2\bar{\nu} / 4h^4(u^{j+1}_{i+1} - u^{j+1}_{i-1})^2 + \bar{\in} / h^2(\Delta t)^2 \quad (19)$$

At i=N-1, i+1=N, i-1=N-2 &

$$\begin{aligned} R_{N-1} = {} & 1/2\Delta t(u^{j+1}_{N-1} - u^{j}_{N-1}) - 1/h^2[u^{j+1}_{N} - 2u^{j+1}_{N-1} - u^{j+1}_{N-2}] \\ & - \bar{\beta} /2h^2\Delta t \,[(u^{j+1}_{N} - 2u^{j+1}_{N-1} + u^{j+1}_{N-2}) - ([u^{j-1}_{N} - 2u^{j-1}_{N-1} - u^{j-1}_{N-2})] \\ & - \bar{\nu} /4h^4[(u^{j+1}_{N} - u^{j+1}_{N-2})^2 (u^{j+1}_{N} - 2u^{j+1}_{N-1} - u^{j+1}_{N-2})] - \bar{\in} / h^2(\Delta t)^2 \\ & [(u^{j+1}_{N} - 2u^{j+1}_{N-1} + u^{j+1}_{N-2}) + (u^{j-1}_{N} - 2u^{j-1}_{N-1} + u^{j-1}_{N-2}) \end{aligned} \quad (20)$$

$$\partial R_{N-1}/ \partial u^{J+1}_{N-1} = 1/2\Delta t + 2/h^2 + 2\bar{\beta}/2h^2\Delta t + 2\bar{\nu}/4h^4 (u^{j+1}_{N} - u^{j+1}_{N-2})^2 + 2\bar{\in} / h^2(\Delta t)^2 \quad (21)$$

$$\partial R_{N-1}/ \partial u^{J+1}_{N} = -1/h^2 - \bar{\beta} /2h^2\Delta t - \bar{\nu} /4h^4 (u^{j+1}_{N} - u^{j+1}_{N-2})^2 - 2\bar{\nu} / 4h^4(u^{j+1}_{N} - u^{j+1}_{N-2}) -(u^{j+1}_{N} - 2u^{j+1}_{N-1} + u^{j+1}_{N-2}) - \in / \bar{h}^2(\Delta t)^2 \quad (22)$$

$$\partial R_{N-1}/ \partial u^{J+1}_{N-2} = -1/h^2 - \bar{\beta} / 2h^2\Delta t - \bar{\nu} /4h^4 (u^{j+1}_{N} - u^{j+1}_{N-2})^2 - 2\bar{\nu} /4h^4(u^{j+1}_{N} - u^{j+1}_{N-2}) \mathbf{x} (u^{j+1}_{N-2} - 2u^{j+1}_{N-1} + u^{j+1}_{N-2}) - \bar{\in} / h^2(\Delta t)^2 \quad (23)$$

4. Results and Discussion

The flow and heat transfer of a third grade fluid past an infinite vertical plate has been studied.

Fig 1 reflects the variation or velocity for different values of the parameter y. It is observed from the figure that velocity increases with increase in y and decreases with decrease of y. For smaller value of y and the larger value of y rapid change in velocity is observed than the intermediate values.

In Fig. 2 and 3 the effect of the parameter α for two different values of y =0.5 and y =0 is depicted. It is observed that in both the cases as α increases the velocity increases & vice versa.

Fig 4 show that the effect of the parameter t on the velocity field for fixed α,y,a,R_e & G_r. It is observed that velocity increases with increase of time & decreases of times.

Fig 5 show the variation of velocity for different values of Re. It is observed that for fixed value of α,y,a & G_r. as R_e increases, the velocity decreases & as R_e decreases, the velocity increases.

In Fig 6 we have studied the effect of the frequency parameter ω on the velocity where the plate oscillates. As ω increases the velocity profile increases

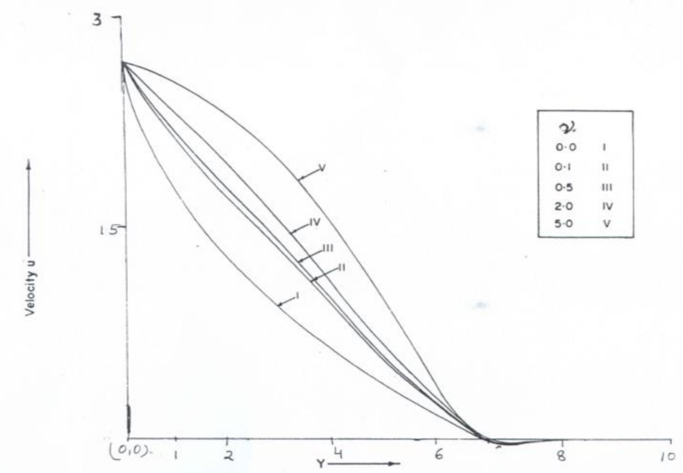

Fig. 1: Velocity distribution for different values of ν for the case t=2, α=-2, a=0.5, Re=1, Gr=5

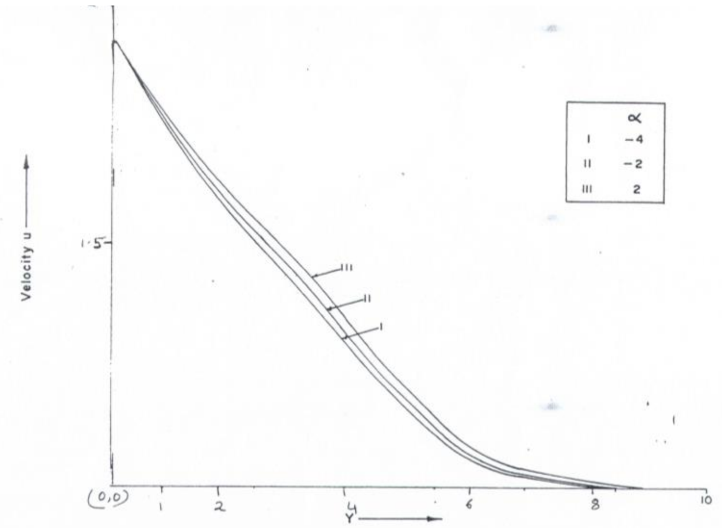

Fig. 2: Velocity distribution for different values of α for the case t =2, a = 0.5, r =-5, Re=1, Gr=5

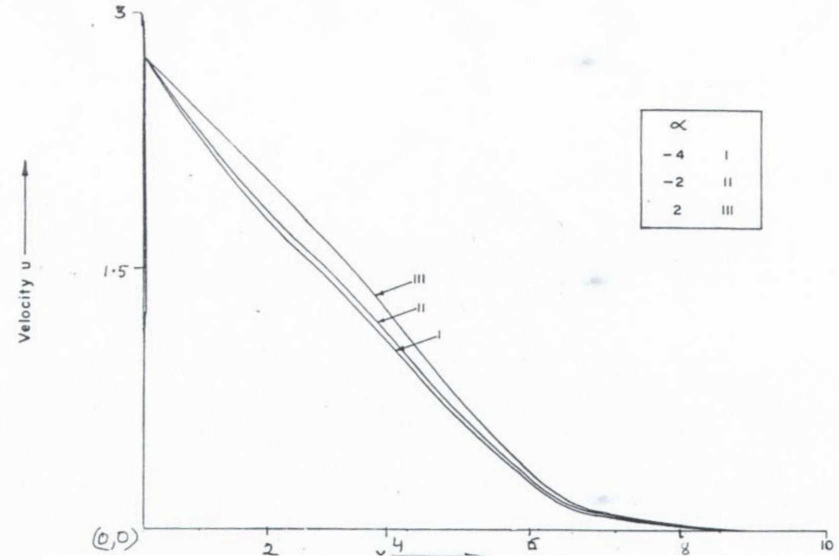

Fig. 3: Velocity distribution for different values of α for the case t=2, a =0.5, r =0, Re=1, Gr =5

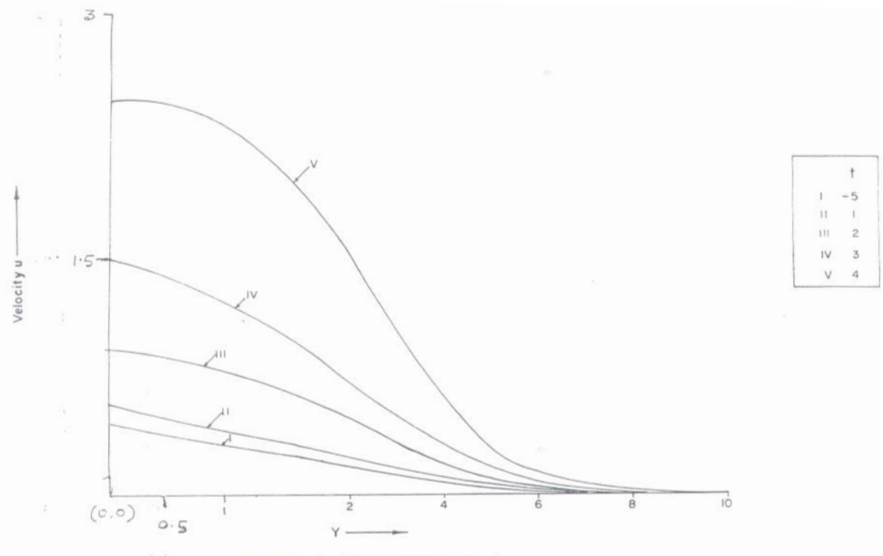

Fig. 4: Velocity distribution for different values of t for the case α =2, r = -5, a = 0.5, Re=1, Gr =5

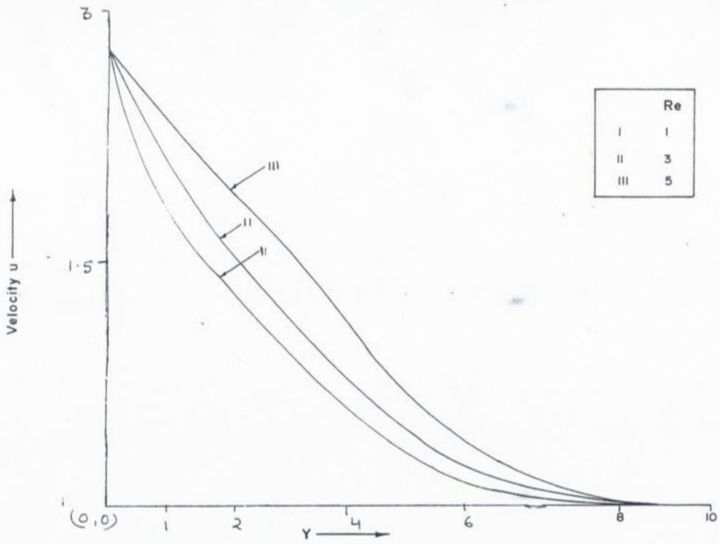

Fig. 5: Velocity distribution for different values of Re for the case $r = .5$, $\alpha = 2$, $a = 0.5$, $Gr = 5$

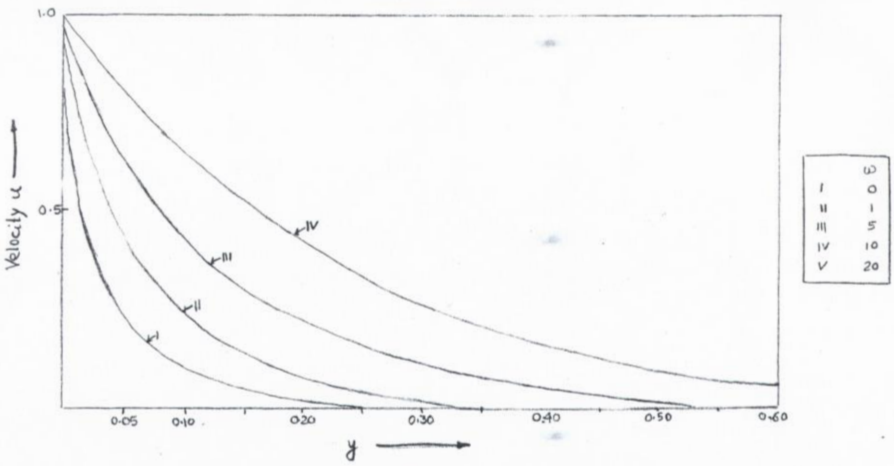

Fig. 6: Velocity distribution for different values of of frequency parameters ω for the case $t = 2$, $a=0.5$, $\nu =0.5$, $Re =1$, $Gr = 5$

References

[1] Joseph et. al, *Arch. Rat. Mech. Anal*. **49,** 321 (1973)

[2] A S Beavers and D.O Joseph, The rotating rod viscometer, J. Fluid Mech.69,475-511 of a second grade fluid for steady and unsteady cases, Acta Mech, 94, 96 (1975)

[3] R L Fosdick and K R Rajgopal, 1980, *Proc.Roy. Soc. London Ser*. **A339**, 351 (1980)

[4] K R Rajagopal and A S Gupta, *Int. J. Eng Sc*. **19**, 1014 (1981)

[5] I Teipcel, *Acta Mech*. **39**, 277 (1981)

[6] K R Rajagopal and T V Na, *Acta Mechanica.* **18**, 179 (1983)

[7] K R Rajgopal and A S Gupta, *Acta. Mechanica*. **19**, 158 (1984)

[8] Y K Garg and K R Rajagopal, *Acta. Mech.* **88**, 113 (1991)

[9] A M Benhabit and A M Siddiqui, *Acta. Mech*. **94**, 96 (1992)

[10] M E Erdogan, *Acta. Mech*. **108**, 179 (1995).

[11] C H Hus and K L Hsiao, *Int. J. Heat Mass Transfer* **41**, 1087 (1998)

[12] Hayat et. Al,. (2001)

[13] T K Hayat, A H Kara and E Iv1omonint, *Int. J. Non-Linear Mechanics* **38**, 1533 (2003)

[14] K Fakhar, Z henil and Yi Cheng, *Exact solution for an unsteady flow of an incompressible fluid of Third Grade* (2008).

[15] O D Makinde & T Chinyoka, *Unsteady Hydromagnetic generalized coutte flow & heat transfer characteristics of a reactive variable viscocity incompressible electrically conducting third grade fluid* (2011).

[16] R Ellahi, *Convergence of series solution of Non-newtonian Third grade fluid with variable viscocity* (2012)

[17] Y N Aiyesimi, O G T Okeday and O W Lawal, Unsteady MHD thin film flow of a Third grade fluid with heat transfer and no slip boundary condition down in an inclined plane (2013)